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RADIALLY SYMMETRIC INTERNAL LAYERS IN A SEMILINEAR ELLIPTIC SYSTEM

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ABSTRACT. Let B denote the unit ball in R^N, $N \geq 1$. We consider the problem of finding nonconstant solutions to a class of elliptic systems including the Gierer and Meinhardt model of biological pattern formation,

$$\varepsilon^2 \Delta u - u + \frac{u^2}{1+ku^2} + \rho = 0 \quad \text{in } B, \tag{1.1}$$

$$D\Delta v - v + u^2 = 0 \quad \text{in } B, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \quad \text{on } \partial B, \tag{1.3}$$

where ε, D, k and ρ denote positive constants and n the unit outer normal to ∂B.

Assuming that the parameters ρ, k are small and D large, we construct a family of radially symmetric solutions to (1.1)–(1.3) indexed by the parameter ε, which exhibits an *internal layer* in B, as $\varepsilon \to 0$.

1. INTRODUCTION

Let B denote the unit ball in R^N, $N \geq 1$. We consider the problem of finding nonconstant solutions to an elliptic system of the form

$$\varepsilon^2 \Delta u = f(u, v) \quad \text{in } B, \tag{1.1}$$

$$D\Delta v = g(u, v) \quad \text{in } B, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \quad \text{on } \partial B, \tag{1.3}$$

where ε and D denote positive constants and n the unit outer normal to ∂B. We are especially interested in identifying a family of solutions to (1.1)–(1.3) indexed by the parameter ε, which exhibits an *internal layer* in B. as $\varepsilon \to 0$. We will refer henceforth to system (1.1)–(1.3) as problem (P).

A good model for the kind of nonlinearities we will consider is given by

$$f(u, v) = u - \frac{u^2}{v(1+ku^2)} + \rho, \tag{1.4}$$

$$g(u, v) = v - u^2, \tag{1.5}$$

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where $k > 0$ and $\rho \geq 0$ are constants. For this nonlinearity, positive solutions of problem (P) correspond to steady states of a reaction-diffusion system proposed by Gierer and Meinhardt [12] as a model of biological pattern formation. Roughly speaking, in that context u and v represent the concentration of two substances, respectively called *activator* and *inhibitor*, ruling a certain chemical process taking place on a piece of tissue represented by the domain. These substances diffuse from cell to cell at respective rates d and D and react chemically in such a way that a relation like (1.1)–(1.3) holds. The Neumann boundary conditions just represent the fact that no diffusion to the exterior occurs.

It is easily checked that, for f and g given by (1.4), (1.5), problem (P) possesses exactly one positive constant solution, the *homogeneous steady state.* On the other hand, it is shown in [8] that (P) possesses only the constant solution in case that the product $\rho k^{1/2}$, for ρ, k in (1.4), is sufficiently large.

Next we state the precise assumptions we will make on f and g throughout this paper. It is not hard to check that they are indeed satisfied by (1.4), (1.5) in case that the parameters k and ρ are sufficiently small.

(H1) f and g are functions defined on some open subset of R^2, f of class C^2, g of class C^1.

(H2) There exists a bounded open interval I such that for all $v \in \overline{I}$ the function $u \mapsto f(u, v)$ possesses exactly three zeros $h_-(v) < h_0(v) < h_+(v)$, two of them nondegenerate and stable, namely

$$f_u(h_\pm(v), v) > 0, \quad \text{for all } v \in \overline{I}.$$

(H3) Set, for $v \in I$,

$$J(v) = \int_{h_-(v)}^{h_+(v)} f(s, v)\, ds.$$

Then there exists a (unique) value $v^* \in I$ such that $J(v^*) = 0$. Moreover, $J'(v^*) \neq 0$.

(H4) For $v \in \overline{I}$ set

$$G_\pm(v) := g(h_\pm(v), v). \tag{1.6}$$

Then

$$G_-(v) > 0 > G_+(v) \quad \text{for all } v \in \overline{I}.$$

Assume the validity of (H1)–(H4). We are interested in nonconstant solutions to problem (P) when ε is small. Fix a number $\theta \in I$ and set

$$h^\theta(v) = \begin{cases} h_-(v) & \text{if } v < \theta, \\ h_+(v) & \text{if } v \geq \theta. \end{cases} \tag{1.7}$$

Setting formally $\varepsilon = 0$ in (1.1), we see that, for a fixed function $v(x)$ whose range lies on I, we can solve for u (1.1) into the form $u(x) = h^\theta(v(x))$. Substituting this u into (1.2) we obtain the boundary value problem, with discontinuous nonlinearity,

$$\Delta v = \sigma g(h^\theta(v), v) = 0 \quad \text{in } B, \qquad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial B. \tag{1.8}$$

Here and henceforth we denote $\sigma := 1/D$.

If we could find a solution $v_0(x)$ to (1.8), then $(u, v) := (h^\theta(v_0), v_0)$ would solve (1.1)–(1.3) for $\varepsilon = 0$. In this situation, it is natural to ask whether we can find a solution $(u_\varepsilon, v_\varepsilon)$ to (P) which is "close" to $(h^\theta(v_0), v_0)$ for ε sufficiently small. For $N = 1$ and Dirichlet boundary conditions, Fife [10] proved that such a family indeed exists if we choose $\theta = v^*$ where v^* is as in (H3), and v_0 is strictly increasing. The method in [10] consists of solving system (P) in two disjoint subintervals of $(0, 1)$ and then matching the solutions in the C^1-sense. Generalized implicit function theorems based on the construction of first approximations to the matching solutions are of assistance in this approach.

Mimura, Tabata and Hosono [19] extended Fife's method to the case of Neumann boundary conditions. They also introduced condition (H4) (plus the assumption $G'_\pm(v) > 0$ for $v \in I$) to construct solutions of problem (1.8).

Subsequent refinements of Fife's method were performed by Ito [15] and Nishiura and Fujii [24]. Sakamoto [25] provided a different construction based upon a first approximation using the idea in [15] and the Lyapunov-Schmidt method. The stability of these solutions is also studied in [15] and [25].

These works have provided us with a good understanding of the so-called families of *layered solutions* to problem (P) in one dimension. However, rather little seems to be known in the higher dimensional case $N > 1$. In related scalar problems and potential systems, higher dimensional layered families have been studied by several authors; see for example [1], [2], [20], [16], [11], [4], [5], [16]. A major technical difficulty arising in the case of system (1.1)–(1.3) is its lack of an obvious variational structure, so that the powerful machinery of the calculus of variations is not directly available here.

In this paper we search for solutions to problem (P) exhibiting radial symmetry. We will establish the existence of a family of radial layered solutions to problem (P) under assumptions (H1)–(H4) provided that $\sigma = 1/D$ is sufficiently small.

The method of construction we will present consists of the following steps:

Step 1. We identify a radially symmetric solution $v_0(|x|)$ to (1.8) for $\theta = v^*$ which takes the value v^* at just one sphere $|x| = \lambda_0$, and is nondegenerate in some appropriate sense.

Step 2. For any fixed radially symmetric v in a small $C^{1,\alpha}$-neighborhood $\mathcal{N}$ of v_0, we solve (1.1) for u into the form $u = k^\varepsilon(v)$, where the operator k^ε satisfies, among other properties,

$$\lim_{\varepsilon\to 0} k^\varepsilon(v) = h^{v^*}(v)$$

uniformly on compacts of $\overline{B}\backslash\{v = v^*\}$.

Step 3. We replace $u = k^\varepsilon(v)$ for $v \in \mathcal{N}$ into (1.2) to obtain the boundary value problem

$$\Delta v = \sigma g(k^\varepsilon(v), v) = 0 \quad \text{in } B, \qquad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial B \tag{1.9}$$

which is a perturbation of (1.8) for $\theta = v^*$ near $v = v_0$. Then we prove the existence of a family of solutions v_ε to (1.9) such that $v_\varepsilon \to v_0$ in the $C^{1,\alpha}$-sense, using a simple degree theoretical argument based on the *nondegeneracy* of v_0 and the properties of k^ε. Hence, $(u_\varepsilon, v_\varepsilon) = (k^\varepsilon(v_\varepsilon), v_\varepsilon)$ is the family of solutions we are looking for. Observe that u_ε exhibits indeed a layered behavior with interface near $\{|x| = \lambda_0\}$.

The method outlined above is natural and seems to be better suited than the matched-solutions approach, to attack higher dimensional situations.

Indeed, our approach in the construction of k^ε seems to apply in a general smooth domain Ω, at the expense of additional technical work, whenever v_0 is a $C^{1,\alpha}$-function such that the level set $\{v_0 = v^*\}$ is a closed $(N-1)$-dimensional hypersurface where ∇v_0 does not vanish. We will elaborate on this matter in a future work.

On the other hand, Step 3 does not require radial symmetry. Instead, the corresponding analogue of Step 1 in a general Ω is more difficult and might require restrictions in its geometry. Basically, one needs to find a solution v_0 to (1.8) as in the above paragraph such that the *linearization* of (1.8) around v_0 in $H^1(\Omega)$ is nonsingular. Problem (1.8) constitutes, without radial symmetry, a nonstandard free-boundary problem which is an interesting mathematical problem in its own right.

The outline of this paper is as follows. In §2 we carry out Step 1, in Propositions 2.1 and 2.2. In §3 we construct the operator $k^\varepsilon(v)$ of Step 2 in Proposition 3.1. Finally, in §4 we state and prove our main result, Theorem 4.1, which carries out Step 3, establishing the existence of the desired family of solutions to (P).

In the remainder of this paper B will always denote the unit ball in R^N and we will assume the validity of assumptions (H1)–(H4). We will use the notations H_r^1, C_r^1, etc., to designate the subspaces of radially symmetric elements of $H^1(B)$, $C^1(\overline{B})$, etc., endowed with their natural norms.

2. Analysis of problem (1.8)

In this section we shall study the problem of finding radial solutions to (1.8). Denote

$$G(v) = \begin{cases} G_-(v) & \text{if } v < \theta, \\ G_+(v) & \text{if } v \geq \theta, \end{cases}$$

where $G_\pm$ are given by (1.6) and θ is a fixed number in I. We consider the problem

$$\Delta v = \sigma G(v) \quad \text{in } B, \qquad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial B. \tag{2.1}$$

By a solution to (2.1) we understand a $v \in C^1(\overline{B})$ satisfying (2.1) in the weak sense. We look for radially symmetric solutions to (2.1), that is, solutions $v = v(r)$ of the boundary value problem

$$v''(r) + \frac{N-1}{r} v'(r) = \sigma G(v(r)), \qquad r \in (0,1), \qquad v'(0) = 0 = v'(1). \tag{2.2}$$

We have the following existence result for a radial solution of (2.1).

Proposition 2.1. *There exists a number $\sigma_0 > 0$ such that for every $\sigma \leq \sigma_0$ (2.1) possesses a radially symmetric solution (2.1) $v_0(r)$ whose range lies on I and such that*

(1) $v_0'(r) < 0$ *for all* $r \in (0, 1)$.

(2) $v_0(\lambda) = \theta$ *at a unique point* $\lambda \in (0, 1)$. *Moreover, for some* $\delta > 0$ *independent of* σ,

$$\delta \le \lambda \le 1 - \delta.$$

Proof. Fix $\lambda \in (0, 1)$ and consider the problems

$$v''(r) + \frac{N-1}{r} v'(r) = \sigma G_+(v(r)), \qquad r \in (0, \lambda), \tag{2.3}$$
$$v'(0) = 0, \quad v(\lambda) = \theta,$$

and

$$v''(r) + \frac{N-1}{r} v'(r) = \sigma G_-(v(r)), \qquad r \in [\lambda, 1), \tag{2.4}$$
$$v'(1) = 0, \quad v(\lambda) = \theta.$$

Without loss of generality assume that $\overline{I}$ is compact. Extend G_+ and G_- to the whole real line in such a way that $G_\pm$ and $G_\pm'$ lie between the same bounds they do on I.

It is easy to see, for example from a direct variational argument, that (2.3) possesses a solution $v_+(r, \lambda)$ such that $v_+'(r, \lambda) < 0$ on $(0, \lambda]$. Moreover, this solution is unique if we ask for σ to be so small that

$$\sigma \sup_I |G_+'| < \mu_1$$

where μ_1 denotes the first eigenvalue of $-\Delta$ on B under Dirichlet boundary conditions.

From (2.3) we immediately see that

$$v_+'(r, \lambda) = \frac{1}{r^{N-1}} \sigma \int_0^r G_+(v) s^{N-1}\, ds \quad \text{for } 0 < r \le \lambda.$$

Hence

$$|v_+'(r, \lambda)| \le \sigma b_+ \lambda \tag{2.5}$$

and

$$|v_+(r, \lambda) - \theta| \le \sigma b_+ \lambda \tag{2.6}$$

where $b_+ := \sup_I |G_+|$. We also have

$$-v_+'(\lambda, \lambda) \ge \sigma a_+ \lambda \tag{2.7}$$

where $a_+ := \inf_I |G_+|$.

Similarly, we can find a solution $v_-(r, \lambda)$ to (2.4) satisfying

$$|v_-'(r, \lambda)| \le \sigma b_- \frac{(1-\lambda^N)}{\lambda^{N-1}} \tag{2.8}$$

and

$$|v_-(r, \lambda) - \theta| \le \sigma b_- \frac{(1-\lambda^N)}{\lambda^{N-1}}. \tag{2.9}$$

Moreover,

$$-v_-'(\lambda, \lambda) \ge \sigma a_- \frac{(1-\lambda^N)}{\lambda^{N-1}}, \tag{2.10}$$

where $a_- := \inf_I |G_-|$.

Denote $\varphi(\lambda) := v'_-(\lambda, \lambda) - v'_+(\lambda, \lambda)$. From inequalities (2.5), (2.7), (2.9) and (2.10) we find that for some $\delta > 0$ sufficiently small $\varphi(\delta) > 0$ and $\varphi(1-\delta) < 0$. Thus, the result of the proposition will follow if we can find a zero of φ on $[\delta, 1-\delta]$. But for $\lambda \geq \delta$, and $\sigma \leq \sigma_0(\delta, G'_-)$ the solution v_- is unique. Indeed, this is a simple consequence of (2.4) and the inequality

$$\int_\lambda^1 h^2(r) r^{N-1}\, dr \leq \frac{1}{\delta^{N-1}} \int_\lambda^1 h'(r)^2 r^{N-1}\, dr$$

satisfied for all $h \in C^1[\lambda, 1]$ with $h(\lambda) = 0$. Thus, if we require σ to be sufficiently small, we have uniqueness for $v_\pm$. From here, the continuity of φ is immediate, so that we obtain the existence of the desired solution. Moreover, inequalities (2.6) and (2.9) imply that the range of this solution is included in I for all sufficiently small σ and the proposition follows. □

Corollary 2.1. *There exists a second radial solution v_1 to (2.1) as in last proposition, but such that $v_1'(r) > 0$ for all $r \in (0, 1)$.*

Proof. Just apply Proposition 2.1 replacing $G(v)$ by $-G(2\theta - v)$. □

Since every function C^1_r-close to v_0 takes the value θ just once, it follows that the operator $\widetilde{G}\colon C^1_r \to L^p_r$ defined by $\widetilde{G}(v)(r) = G(v(r))$ is continuous on a neighborhood of v_0, for all $1 \leq p < \infty$.

Fix a number $p > m$. Then $W^{2,p}_r$ is compactly embedded into $C^{1,\alpha}_r$ for some $\alpha > 0$. Set

$$X := \{v \in C^{1,\alpha}_r \mid v'(1) = 0\} \tag{2.11}$$

endowed with its natural norm. Denote $R = (\Delta - I)^{-1}$ under Neumann boundary conditions, and define the operator $T\colon X \to X$ by

$$T(v) := R(\sigma \widetilde{G}(v) - v). \tag{2.12}$$

Then T is completely continuous on a neighborhood of v_0. Observe that fixed points of T are precisely the radial solutions of (2.1). Our main goal in the remainder of this section is to prove the following result.

Proposition 2.2. *There exists a number $\sigma_0 > 0$ such that, for each fixed $\sigma \leq \sigma_0$, there is an X-neighborhood $\mathcal{N}$ of v_0 such that T does not possess fixed points other than v_0 on $\overline{\mathcal{N}}$ for all $\sigma \leq \sigma_0$ and*

$$\deg(I - T, \mathcal{N}, v_0) \neq 0 \tag{2.13}$$

where T is given by (2.12) and I denotes the identity operator in X.

The fact that this degree is nonzero is a key ingredient in the construction of the family of solutions we are looking for.

To prove (2.13) we will try to linearize the operator T around v_0. This will certainly require some kind of linearization of $\widetilde{G}$. Consider $\widetilde{G}$ as an operator from H^1_r into H^{-1}_r where H^{-1}_r denotes the dual space of H^1_r. Here we identify $\widetilde{G}(v)$ with the functional

$$\phi \in H^1_r \mapsto \int_0^1 G(v)\phi r^{N-1}\, dr.$$

$\widetilde{G}$ defined between these spaces turns out to be Fréchet differentiable at v_0 as we shall show next. Observe, however, that $\widetilde{G}$ is not even continuous on any H_r^1-neighborhood of v_0.

Proposition 2.3. *$\widetilde{G}\colon H_r^1 \to H_r^{-1}$ defined above is Fréchet differentiable at v_0. Its derivative at v_0, $\widetilde{G}'(v_0)$, is the operator L defined by*

(2.14)
$$\langle Lh, \phi\rangle = \int_0^\lambda G_+'(v_0)\phi h r^{N-1}\,dr + \int_\lambda^1 G_-'(v_0)\phi h r^{N-1}\,dr + \eta h(\lambda)\phi(\lambda), \qquad h, \phi \in H_r^1,$$

where λ is the unique point where v_0 takes the value θ and η is the negative constant

$$\eta := \frac{(G_+ - G_-)(\theta)}{-v_0'(\lambda)}\lambda^{N-1}. \tag{2.15}$$

Proof. Let $\phi \in H_r^1$. We can write

$$\begin{aligned}
&\langle \widetilde{G}(v) - \widetilde{G}(v_0) - Lh, \phi\rangle \\
&= \int_{(0,\lambda)\cap\{v>\theta\}} (G_+(v) - G_+(v_0) - G_+'(v_0)(v - v_0))\phi r^{N-1}\,dr \\
&+ \int_{(\lambda,1)\cap\{v<\theta\}} (G_-(v) - G_-(v_0) - G_-'(v_0)(v - v_0))\phi r^{N-1}\,dr \\
&+ \int_{(\lambda,1)\cap\{v>\theta\}} (G_+(v) - G_-(v_0) - G_-'(v_0)(v - v_0))\phi r^{N-1}\,dr \\
&+ \int_{(0,\lambda)\cap\{v<\theta\}} (G_-(v) - G_+(v_0) - G_+'(v_0)(v - v_0))\phi r^{N-1}\,dr \\
&- \eta(v - v_0)(\lambda)\phi(\lambda) + \int_{\{v=\theta\}} f(r)\phi(r) r^{N-1}\,dr
\end{aligned} \tag{2.16}$$

where f satisfies

$$|f(r)| \le c, \qquad r \in (0, 1), \tag{2.17}$$

where c is a certain constant independent of v. Let us first estimate the last integral in (2.16). To do this, we will first estimate the measure of the set $\{v = \theta\}$, $|\{v = \theta\}| = \int_{\{v=\theta\}} r^{N-1}\,dr$.

Since $v \in H_r^1$, we have that $v' = 0$ almost everywhere in $\{v = \theta\}$. Fix $\delta > 0$. Then $-v_0' \ge c(\delta) > 0$ on $(\delta, 1-\delta)$ and

$$\int_{\{v=\theta\}\cap(\delta,1-\delta)} (-v_0')\,dr = \int_{\{v=\theta\}\cap(\delta,1-\delta)} (v - v_0)'\,dr$$

so that

$$\begin{aligned}
&\frac{c(\delta)}{(1-\delta)^{N-1}}|\{v = \theta\}\cap(\delta, 1-\delta)| \\
&\le \frac{1}{\delta^{N-1}}\left(\int_0^1 (v - v_0)'^2 r^{N-1}\,dr\right)^{\frac{1}{2}} |\{v = \theta\}\cap(\delta, 1-\delta)|^{\frac{1}{2}};
\end{aligned}$$

hence

$$|\{v = \theta\} \cap (\delta, 1-\delta)| \le k(\delta)\|v - v_0\|^2_{H^1_r}. \tag{2.18}$$

On the other hand, set

$$\mu := \inf_{0<r<\delta}(v_0(r) - \theta) > 0.$$

Then Tchebyshev's inequality yields

$$|\{v = \theta\} \cap (0, \delta)| \le |\{|v - v_0| > \mu\}| \le \frac{1}{\mu^2}\|v - v_0\|^2_{L^2_r}.$$

A similar estimate holds for $|\{v = \theta\} \cap (1-\delta, 1)|$. From this and (2.18) we conclude that

$$|\{v = \theta\}| \le c\|v - v_0\|^2_{H^1_r}.$$

Next, fix some $q > 2$ such that H^1_r is continuously embedded in L^q_r. Then, using (2.17),

$$\begin{aligned}\left|\int_{\{v=\theta\}} f(r)\phi(r)r^{N-1}\,dr\right| &\le c\|\phi\|_{L^q_r}|\{v = \theta\}|^{1/q'}\\ &\le c\|v - v_0\|^{2/q}_{H^1_r}\|\phi\|_{L^q_r} = \|\phi\|_{H^1_r}o(\|v - v_0\|_{H^1_r}).\end{aligned}$$

It remains to obtain similar estimates for the rest of the integrals in (2.16). We begin with the first two, which we call respectively I and II. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that for all $r \in [0, 1]$ and all t such that $|t - v_0(r)| < \delta$ one has

$$|(G_+(t) - G_+(v_0(r)) - G'_+(v_0(r))(t - v_0(r))| \le \varepsilon|t - v_0(r)|.$$

We estimate I as follows.

$$\begin{aligned}|\mathrm{I}| &\le \int_{\{|v-v_0|<\delta\}} + \int_{\{|v-v_0|\ge\delta\}} |(G_+(v) - G_+(v_0) - G'_+(v_0)(v - v_0))|\,|\phi|r^{N-1}\,dr\\ &\equiv \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

Then

$$\mathrm{I}_1 \le C\varepsilon\|v - v_0\|_{L^2}\|\phi\|_{L^2}$$

and, for a fixed, small $a > 0$,

$$\mathrm{I}_2 \le C\|v - v_0\|_{L^{2+a}}\|\phi\|_{L^{2+a}}|\{|v - v_0| > \delta\}|^{1-2/(2+a)}$$

where we have used the fact that G is Lipschitz. Since

$$|\{|v - v_0| > \delta\}| \le \frac{1}{\delta^2}\|v - v_0\|^2_{L^2},$$

we conclude after combining the above estimates and choosing conveniently a,

$$\mathrm{I} \le \|\phi\|_{H^1_r}o(\|v - v_0\|_{H^1_r}).$$

A similar estimate of course holds true for II. Let us call III and IV respectively the third and fourth integrals in the decomposition (2.16). We will show that

$$|\mathrm{III} + \mathrm{IV} - \eta(v - v_0)(\lambda)\phi(\lambda)| \le \|\phi\|_{H^1_r}o(\|v - v_0\|_{H^1_r}).$$

Denote $A_1 := (0, \lambda) \cap \{v < \theta\}$ and $A_2 := (\lambda, 1) \cap \{v \geq \theta\}$. We will first estimate the measure of these sets.

Assume $v(\lambda) - \theta > 0$ and let $\lambda' > \lambda$ be the first point where $v(\lambda') - \theta = 0$ (observe that there must be such a point provided that $\|v - v_0\|_{H_r^1}$ is sufficiently small). We will estimate the size of the interval (λ, λ').

We have that

$$v(\lambda) - v_0(\lambda) = v(\lambda) - \theta = \int_\lambda^{\lambda'} (v_0 - v)' + \int_\lambda^{\lambda'} v_0'. \tag{2.19}$$

Without any loss of generality we may assume that λ' is away from 1. Then v_0' is away from zero on (λ, λ'). Then (2.19) implies

$$(\lambda - \lambda') \leq c \left\{ |v(\lambda) - v_0(\lambda)| + \frac{1}{\lambda^{N-1}} \left\{ \int_\lambda^{\lambda'} (v - v_0)'^2 r^{N-1}\, dr \right\}^{\frac{1}{2}} \right\} \leq c\|v - v_0\|_{H_r^1}. \tag{2.20}$$

On the other hand, combining (2.19) and (2.20) yields the estimate

$$\lambda' - \lambda = \frac{v(\lambda) - \theta}{-v_0'(\lambda)} + o(\|v - v_0\|_{H_r^1}). \tag{2.21}$$

We next estimate the size of the rest of A_1. Set $\widetilde{A}_1 = A \backslash [\lambda, \lambda']$. Since $v(1) < 0$ for $\|v - v_0\|_{H_r^1}$ sufficiently small, we obtain, after writing $(\lambda_j, \lambda_{j+1})$, $j = 1, 2, \ldots,$ for the components of $\widetilde{A}_1$, that $v(\lambda_j) = \theta$ for all j. Then

$$\int_{\widetilde{A}_1} v' = \sum_j \int_{\lambda_j}^{\lambda_{j+1}} v' = 0.$$

Hence

$$c|\widetilde{A}_1| \leq -\int_{\widetilde{A}_1} v_0' = -\int_{\widetilde{A}_1} (v_0' - v') \leq c|\widetilde{A}_1|^{\frac{1}{2}} \|v - v_0\|_{H_r^1}$$

since we may assume v_0' away from zero on $\widetilde{A}_1$. Then

$$|\widetilde{A}_1| \leq c\|v - v_0\|^2_{H_r^1}.$$

A similar estimate works for A_2, except that A_2 may not be bounded below away from zero. But for a fixed $0 < \delta < \lambda$ we may decompose

$$A_2 = (A_2 \cap [0, \delta]) \cup (A_2 \cap [\delta, \lambda)).$$

Applying Tchebyshev's inequality as in the estimate for $|\{v = \theta\}|$ yields

$$|A_2 \cap [0, \delta]| \leq c\|v - v_0\|^2_{L_r^2}.$$

Using this and the same argument we employed to estimate $\widetilde{A}_1$ we obtain

$$|A_2| \leq c\|v - v_0\|^2_{H_r^1}.$$

Thus, we have shown

$$|A_1 \backslash [\lambda, \lambda']| + |A_2| = o(\|v - v_0\|_{H_r^1}). \tag{2.22}$$

From here, we immediately find an estimate for integral IV in (2.16). In fact,

$$\mathrm{IV} = \int_{A_2} (G_+(v) - G_-(v))\phi r^{N-1}\, dr + \int_{A_2} (G_-(v) - G_-(v) - G'_-(v_0)(v - v_0))\phi r^{N-1}\, dr.$$

Estimating the second integral in the above expression as we did with I, and the first integral using (2.22) we obtain

$$|\mathrm{IV}| \le \|\phi\|_{H^1_r} o(\|v - v_0\|_{H^1_r}).$$

A similar estimate holds for the part of integral III outside (λ, λ'). Therefore, it only remains to estimate the quantity

$$\begin{aligned} J := &\int_\lambda^{\lambda'} (G_+(v) - G_+(v_0))\phi r^{N-1}\, dr \\ &+ \int_\lambda^{\lambda'} ((G_+ - G_-)(v_0) - (G_+ - G_-(v_0))(\theta))\phi r^{N-1}\, dr \\ &- \int_\lambda^{\lambda'} G'_-(v_0)(v - v_0)\phi r^{N-1}\, dr \\ &+ \{G_+(\theta) - G_-(\theta)\}\left\{\int_\lambda^{\lambda'} \phi r^{N-1}\, dr - \lambda^{N-1}\frac{(v(\lambda) - \theta)}{-v_0'(\lambda)}\phi(\lambda)\right\}. \end{aligned} \tag{2.23}$$

Since (λ, λ') is away from zero, we have

$$\sup_{(\lambda, \lambda')} |\phi| \le c\|\phi\|_{H^1_r}.$$

Using this, the fact that $G_\pm$ may be assumed to be Lipschitz and $|\lambda - \lambda'| \le c\|v - v_0\|_{H^1_r}$ we easily derive an estimate of the form $\|\phi\|_{L^2_r} o(\|v - v_0\|_{H^1_r})$ for the first three integrals in (2.23). It remains to estimate the last part. From (2.21) we obtain

$$\int_\lambda^{\lambda'} \phi r^{N-1}\, dr - \lambda^{N-1}\frac{(v(\lambda) - \theta)}{-v_0'(\lambda)}\phi(\lambda) = \int_\lambda^{\lambda'} \phi(r^{N-1} - \lambda^{N-1})\, dr + \lambda^{N-1}\int_\lambda^{\lambda'} (\phi(r) - \phi(\lambda))r^{N-1}\, dr + \phi(\lambda)o(\|v - v_0\|_{H^1_r}).$$

But

$$\left|\int_\lambda^{\lambda'} \phi(r^{N-1} - \lambda^{N-1})\, dr\right| \le c(\lambda' - \lambda) \sup_{r\in(\lambda, \lambda')} |\phi(r) - \phi(\lambda)| \le c(\lambda' - \lambda)\left(\int_\lambda^{\lambda'} \phi'(r)^2 r^{N-1}\, dr\right)^{\frac{1}{2}} \le \|\phi\|_{H^1_r} o(\|v - v_0\|_{H^1_r}).$$

Combining all these estimates we finally obtain the validity of an inequality of the form

$$\langle \widetilde{G}(v) - \widetilde{G}(v_0) - Lh, \phi\rangle \le \|\phi\|_{H^1_r} o(\|v - v_0\|_{H^1_r})$$

which gives the desired result. Recall that we assumed $v(\lambda)-\theta>0$. The case $v(\lambda)-\theta<0$ is similar. If $v(\lambda)-\theta=0$ it is even easier since in that case we find

$$|A_1|+|A_2|=o(\|v-v_0\|_{H^1_r}).$$

This concludes the proof. □

Remark 2.1. $\widetilde{G}'(v_0)\colon H^1_r\to H^{-1}_r$ is a compact operator. Indeed, let h_n be a bounded sequence in H^1_r. Then, passing to a subsequence which we still denote h_n, we may assume $h_n\rightharpoonup h$ in H^1_r weakly, hence strongly in L^2_r and uniformly on compacts of $(0,1]$. In particular, $h_n(\lambda)\to h(\lambda)$. Then, from (2.14),

$$|\langle\widetilde{G}'(v_0)[h_n-h],\phi\rangle|\le c\|h_n-h\|_{L^2_r}+|\eta|\,|h_n(\lambda)-h(\lambda)|\,|\phi(\lambda)|\le\|\phi\|_{H^1_r}o(1).$$

Thus, $\widetilde{G}'(v_0)h_n\to\widetilde{G}'(v_0)h$ strongly, which proves the remark. We know that $R=(\Delta-I)^{-1}$ under Neumann boundary conditions is a linear and continuous operator from H^{-1}_r into H^1_r. It follows from the above remark that the operator $S\colon H^1_r\to H^1_r$ defined as

$$S(h):=R(\sigma\widetilde{G}'(v_0)h-h) \tag{2.24}$$

is compact. Observe that $(I-S)$ can be interpreted as the "linearization" of $(I-T)$ around v_0 with T defined by (2.12). Since we are interested in computing the local degree of $(I-T)$ around v_0 in X, it seems to be natural to study the degree of $(I-S)$ around zero in H^1_r.

Lemma 2.1. *The operator $(I-S)$, where S is defined by (2.24), is a linear isomorphism of H^1_r, provided that σ is sufficiently small.*

Proof. Since S is compact, it suffices to show that $(I-S)$ has trivial kernel. Let $\phi\in H^1_r$ satisfy $(I-S)\phi=0$. Then ϕ satisfies in the distributional sense

$$\phi''(r)+\frac{N-1}{r}\phi'(r)=\sigma\alpha(r)\phi(r)+\sigma\eta\phi(\lambda)\delta_\lambda(r),\qquad r\in(0,1), \tag{2.25}$$
$$\phi'(0)=0=\phi'(1)$$

where $\delta_\lambda(r)$ denotes the Dirac delta supported at λ, η is given by (2.15) and

$$\alpha(r)=\begin{cases}G'_+(v_0(r)) & \text{if } r<\lambda,\\ G'_-(v_0(r)) & \text{if } r\ge\lambda\end{cases} \tag{2.26}$$

so that ϕ satisfies in the classical sense

$$\phi''(r)+\frac{N-1}{r}\phi'(r)=\sigma G'_+(v_0(r))\phi(r),\quad 0<r<\lambda, \tag{2.27}$$

$$\phi''(r)+\frac{N-1}{r}\phi'(r)=\sigma G'_-(v_0(r))\phi(r),\qquad \lambda<r<1, \tag{2.28}$$
$$\phi'(0)=0=\phi'(1)$$

and

$$\phi'(\lambda_+)-\phi'(\lambda_-)=\eta\phi(\lambda). \tag{2.29}$$

Assume $\phi\not\equiv0$. We claim that ϕ never vanishes on $[0,1]$. Indeed, assume for instance that there is a $\lambda_0\le\lambda$ such that $\phi(\lambda_0)=0$. Then,

$$\mu_1\int_0^{\lambda_0}|\phi|^2r^{N-1}\,dr\le\int_0^{\lambda_0}|\phi'|^2r^{N-1}\,dr=\sigma\int_0^{\lambda_0}G'_+(v_0)|\phi|^2r^{N-1}\,dr$$

where μ_1 denotes the first eigenvalue of $-\Delta$ in B. Hence, if σ is sufficiently small we obtain that $\phi \equiv 0$ on $[0, \lambda_0]$, hence on $[0, \lambda]$. Now, recall from Proposition 2.1 that $\lambda \geq \delta$ for some $\delta > 0$ independent of σ. Since $\phi(\lambda) = 0$, there exists a positive constant $\mu' = \mu(\delta)$ such that

$$\mu' \int_\lambda^1 |\phi|^2 r^{N-1}\, dr \leq \int_\lambda^1 |\phi'|^2 r^{N-1}\, dr$$

from which it follows that $\phi \equiv 0$ on $[\lambda, 1]$; hence $\phi \equiv 0$ on $[0, 1]$, which proves the claim.

Thus, we can define

$$w(r) := r^{N-1} \frac{\phi'(r)}{\phi(r)}.$$

From (2.27), (2.28), it is easily verified that $w(r)$ satisfies the equation

$$w' + \frac{w^2}{r^{N-1}} = \sigma\alpha(r) r^{N-1} \tag{2.30}$$

for $r \neq \lambda$. Also, $w(0) = w(1) = 0$. Assume there is a point r_0 where w maximizes on $(0, \lambda]$. Then we must have $w'(r_0) \geq 0$, and hence (2.30) implies

$$w^2(r_0) \leq c\sigma\lambda^{2(N-1)}$$

for some $c > 0$. In particular,

$$w(\lambda^-) \leq c\sqrt{\sigma}\lambda^{N-1}.$$

If w maximizes at 0 on $[0, \lambda]$, the above inequality trivially holds. A similar argument shows that, also,

$$-w(\lambda^+) \leq c\sqrt{\sigma}\lambda^{N-1}.$$

Hence

$$\frac{\phi'(\lambda^-) - \phi'(\lambda^+)}{\phi(\lambda)} \leq c\sqrt{\sigma}. \tag{2.31}$$

But from (2.29), (2.31) and the definition of η in (2.15) we find

$$\frac{\sigma}{-v_0'(\lambda)}(G_- - G_+)(\theta)\lambda^{N-1} \leq c\sqrt{\sigma}. \tag{2.32}$$

But $\lambda \geq \delta > 0$ and, from (2.25), (2.27), $-v_0'(\lambda) \leq c\sigma$; hence the left-hand side of (2.32) is bounded below away from zero. We have reached a contradiction in case that σ is sufficiently small. This concludes the proof. □

We will need for the proof of Proposition 2.2 the following approximation lemma.

Lemma 2.2. *Let D be a bounded and smooth domain in R^N and set $R := (\Delta - I)^{-1}$ under Neumann boundary conditions on D. Denote $H^{-1}(D)$ for the dual space of $H^1(D)$. Fix a number $p > 1$. Then there exists a sequence of linear operators $R_n \colon H^{-1} \to W^{2,p}$ with finite dimensional ranges contained in $R(L^p)$ and satisfying the following properties:*

(1) *$\lim_{n\to\infty} R_n y = Ry$ in the C^1-sense for each $y \in L^p$. Also,*

$$\sup_{n \in N} \|R_n\|_{\mathscr{L}(L^p, W^{2,p})} < +\infty.$$

(2) $\lim_{n\to\infty} R_n y = Ry$ *in the* H^1*-sense for each* $y \in H^{-1}$. *Also,*

$$\sup_{n\in N} \|R_n\|_{\mathscr{L}(H^{-1}, H^1)} < +\infty.$$

Moreover, if D *is an annulus or a ball, and* y *is radially symmetric, then* $R_n y$ *can be chosen radially symmetric*

Proof. In [18, Chapter 2], an orthonormal basis of $L^2(R^N)$ is constructed of the form $\{\phi(x-k)\}_{k\in Z^N}$ where ϕ is in the Schwarz space of rapidly decreasing functions. Moreover, the associated orthonormal projections can be extended to other functional spaces such as $L^p(R^N)$ or $H^{-1}(R^N)$. More precisely, for $y \in L^p(R^N)$ (resp. $y \in H^{-1}(R^N)$) one has

$$y_n := \sum_{|k|\le n} \langle y, \phi_k\rangle \phi_k \to y \quad \text{as } n \to \infty$$

in the sense of $L^p(R^N)$ (resp. $H^{-1}(R^N)$), where $\phi_k(x) = \phi(x-k)$, $\langle y, \phi_k\rangle = \int y\phi_k$ (resp. $\langle y, \phi_k\rangle_{H^{-1}}$). It is also shown in [18] that for $y \in L^p(R^N)$ one has

$$\|y_n\|_{L^p(R^N)} \le c\|y\|_{L^p(R^N)}.$$

Next, we define y_n for $y \in H^{-1}(D)$ as

$$y_n := \sum_{|k|\le n} \langle y, \phi_k|_D\rangle \phi_k|_D.$$

Observe that $\tilde{y}$ defined by $\langle \tilde{y}, \zeta\rangle := \langle y, \zeta|_D\rangle$ is in $H^{-1}(R^N)$; hence we still have that $y_n \to y$ in the $H^{-1}(D)$-sense. Similar statements hold for $y \in L^p(D)$. We observe next that, for $y \in H^{-1}(D)$,

$$\|y_n\|_{H^{-1}(D)} \le c\|y\|_{H^{-1}(D)}. \tag{2.33}$$

Indeed, let $\zeta \in H^1(D)$. Then

$$\begin{aligned} |\langle y_n, \zeta\rangle| &\le \sum_{|k|\le n} |\langle y, \phi_k|_D\rangle| \int_D |\phi_k \zeta|\, dx \\ &\le \|y\|_{H^{-1}(D)} \|\phi\|_{H^1(R^N)} \sum_k \int_D |\phi(x-k)|\, |\zeta(x)|\, dx. \end{aligned} \tag{2.34}$$

Since ϕ is rapidly decreasing, we have

$$|\phi(x)| \le \frac{c}{(1+|x|^2)^N}.$$

It follows that

$$\begin{aligned} \sum_k \int_D |\phi(x-k)|\, |\zeta(x)|\, dx &\le c \sum_k \frac{1}{(1+|k|^2)^N} \int_D |\zeta(x)|\, dx \\ &\le M\|\zeta\|_{H^1(D)}. \end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35), (2.33) follows.

We next define the operators R_n. For $y \in H^{-1}(D)$, define $w_n = R_n y$ to be the unique solution of $(\Delta - I)w = y_n$ under Neumann boundary conditions. In other words, $R_n y := R y_n$. Standard elliptic estimates imply that

(a) If $y \in H^{-1}(D)$, then $w_n \to Ry$ in $H^1(D)$, and

(b) If $y \in L^p$, $p > 1$, then $w_n \to Ry$ in $W^{2,p}(D)$.
Moreover, from the above estimates we also have that $\|R_n\|_{\mathscr{L}(H^{-1}, H^1)}$ and $\|R_n\|_{\mathscr{L}(L^p, W^{2,p})}$ are uniformly bounded, as desired. Finally, in the radially symmetric case, observe that $R_n y$ is not necessarily radial if y is. But in this case we can replace R_n by QR_n where for $w \in H^1(D)$ or $w \in W^{2,p}(D)$

$$(Qw)(r) := \int_{|\xi|=1} w(r\xi)\, d\sigma(\xi)$$

and the integrand is understood in the sense of traces. □

Our last preliminary to the proof of Proposition 2.2 is the following simple result.

Lemma 2.3. *Let R_n be as in Lemma 2.2 for $D = B$. Define $S_n : H_r^1 \to H_r^1$ as*

$$S_n w := R_n(\sigma \widetilde{G}'(v_0)w - w) \tag{2.36}$$

with $\sigma \le \sigma_0$, σ_0 given by Lemma 2.1. Then there exists a $c > 0$ such that for every sufficiently large n one has

$$\|(I - S_n)w\|_{H_r^1} \ge c\|w\|_{H_r^1}$$

for all $w \in H_r^1$.

Proof. Assume the contrary. Then there exists a sequence w_n such that $\|w_n\|_{H_r^1} = 1$ and

$$\|(I - S_n)w_n\|_{H_r^1} \to 0 \quad \text{as } n \to \infty. \tag{2.37}$$

Since the operator $H_r^1 \to H_r^{-1}$, $w \mapsto \sigma\widetilde{G}'(v_0)w - w$ is compact, we may assume $\sigma\widetilde{G}'(v_0)w_n - w_n \to z$ in H_r^{-1}. Now,

$$S_n w_n - Rz = R_n(\sigma\widetilde{G}'(v_0)w_n - w_n - z) + (R_n z - Rz).$$

The second term of the right-hand side of the above expression tends to zero by the last lemma. The first one also does since $\|R_n\|_{\mathscr{L}(H^{-1}, H^1)}$ is uniformly bounded. Hence $S_n w_n \to Rz \equiv w_0$. Then (2.37) implies $w_n \to w_0$.

Finally, if S denotes the operator defined by (2.24), we obtain

$$\begin{aligned} S_n w_n - S w_0 = {} & R_n(\sigma\widetilde{G}'(v_0)(w_n - w_0) - (w_n - w_0)) \\ & + (R_n - R)(\sigma\widetilde{G}'(v_0)w_0 - w_0) \end{aligned}$$

and this expression is easily seen to approach zero as $n \to \infty$. It follows from (2.37) that $(I - S)w_0 = 0$. But since $\|w_0\|_{H_r^1} = 1$, this contradicts the fact that $(I - S)$ is an isomorphism, concluding the proof. □

Proof of Proposition 2.2. We want to show that

$$\deg_X(I - T, \mathscr{N}, v_0) \ne 0$$

where $\mathscr{N}$ is a sufficiently small neighborhood of v_0 in the space X defined by (2.11). Equivalently, we need to show that for δ sufficiently small the degree

$$d := \deg_X(w - \sigma R(\widetilde{G}(v_0 + w) - \widetilde{G}(v_0)), B(0, \delta), 0) \tag{2.38}$$

is well defined and nonzero. Let us accept that this degree is well defined for some small δ. We will prove this fact later. We claim that, for all n sufficiently large,

$$d = \deg_{E_n}(w - \sigma R_n(\widetilde{G}(v_0 + w) - \widetilde{G}(v_0)), B(0, \delta) \cap E_n, 0) \tag{2.39}$$

where R_n is the operator given by Lemma 2.2 for $p = q$, E_n its range. Observe that $E_n \subset X$. Set

$$\varepsilon = \inf_{\|w\|_X = \delta} \|w - \sigma R(\widetilde{G}(v_0 + w) - \widetilde{G}(v_0))\|_X.$$

This number is positive since we are assuming d is well defined. From the definition of the degree (see e.g. Deimling [6, p. 57]), we know that for (2.39) to hold it suffices to show that

$$\sigma \sup_{\|w\|_X \leq \delta} \|(R - R_n)(\widetilde{G}(v_0 + w) - \widetilde{G}(v_0))\|_X < \varepsilon. \tag{2.40}$$

We will see that (2.40) holds true for all n sufficiently large. Assume the contrary. Then there is a sequence w_n with $\|w_n\|_X \leq \delta$ such that

$$\sigma \|(R - R_n)(\widetilde{G}(v_0 + w_n) - \widetilde{G}(v_0))\|_X \geq \varepsilon. \tag{2.41}$$

Ascoli's Theorem implies that we may assume w_n converges uniformly. Hence $\sigma G(v_0 + w_n)$ converges in L^p_r to some $z \in L^p_r$ for any given $p \geq 1$. Thus

$$\begin{aligned} \varepsilon \leq \|R(\sigma \widetilde{G}(v_0 + w_n) - z)\|_X + \|(R - R_n) z\|_X \\ + \|R_n(\sigma \widetilde{G}(v_0 + w_n) - z)\|_X + \sigma \|(R - R_n) G(v_0)\|_X, \end{aligned}$$

but each of these terms tends to zero thanks to the continuity of R and the first part of Lemma 2.3. Hence (2.41) is impossible and (2.40) holds for all n sufficiently large.

On the other hand, again the definition of the degree implies that

$$\begin{aligned} d' := \deg_{H^1_r}(I - S, V, 0) &= \deg_{H^1_r}(w - R(\sigma \widetilde{G}'(v_0) w - w), V, 0) \\ &= \deg_{E_n}(w - R_n(\sigma \widetilde{G}'(v_0) w - w), V \cap E_n, 0) \end{aligned}$$

provided that n is so large that

$$\sup_{\|w\|_{H^1_r} = 1} \|(R - R_n)(\sigma \widetilde{G}'(v_0) w - w)\|_{H^1_r} < \inf_{\|w\|_{H^1_r} = 1} \|(I - s) w\|_{H^1_r}.$$

Here V is any neighborhood of 0 in H^1_r. Since E_n is finite dimensional, we can find V so that

$$V \cap E_n = B(0, \delta) \cap E_n := \Lambda_n.$$

Observe that since $(I - S)$ is an isomorphism, the number d' is nonzero. We will show that $d' = d$, which will prove the result. To do this, consider the homotopy in E_n

$$T^n_t w := (1 - t) R_n(\sigma \widetilde{G}(v_0 + w) - \sigma \widetilde{G}(v_0) - w) + t R_n(\sigma \widetilde{G}'(v_0) w - w)$$

for $t \in [0, 1]$. To show $d = d'$ it clearly suffices to verify that $(I - T^n_t) w \neq 0$ for all $w \in \partial \Lambda_n$. Observe that

(2.42)

$$\begin{aligned} &\|(I - T^n_t) w\|_{H^1_r} \\ &\quad \geq \|(I - S_n) w\|_{H^1_r} - \sigma \|R_n\|_{\mathscr{L}(H^{-1}_r, H^1_r)} \|\widetilde{G}(v_0 + w) - \widetilde{G}(v_0) - \widetilde{G}'(v_0) w\|_{H^{-1}} \\ &\quad \geq c_1 \|w\|_{H^1_r} - c_2 \|\theta(w)\|_{H^{-1}_r} \end{aligned}$$

where S_n is defined by (2.36), and the positive constants c_1, c_2 come respectively from Lemmas 2.3 and 2.2. We have also denoted $\theta(w) := G(v_0 + w) - G(v_0) - G'(v_0)w$. Since $\widetilde{G}$ is Fréchet differentiable at v_0, we see from (2.42) that

$$\|(I - T_t^n)w\|_{H_r^1} \geq c\|w\|_{H_r^1} \tag{2.43}$$

provided that $\|w\|_{H_r^1} < \delta$, for some δ sufficiently small, independent of t and n. From here, the desired result follows after choosing, a priori, δ small enough and observing that $\|w\|_{H_r^1} \leq c\|w\|_X$ for some $c > 0$. Incidentally, these estimates also imply that the degree d is well defined: just use (2.43) for $t = 0$ and let $n \to \infty$. This completes the proof. □

Remark 2.2. The result of Proposition 2.2 also holds true for the increasing solution v_1 of Corollary 2.1.

Remark 2.3. The solution v_0 is, in an appropriate weak sense, unstable. Indeed, the "linearized eigenvalue problem" associated to (2.1) at v_0 has its first eigenvalue μ variationally characterized as

$$\mu = \inf_{\phi \in H_r^1,\, \int_B \phi^2 = 1} \left\{ \int_B |\nabla \phi|^2 + \sigma \int_B \alpha \phi^2 + \sigma \eta \phi(\lambda)^2 \right\}.$$

Here, α was defined in (2.26) and η in (2.15). Using the test function $\phi \equiv 1$, we easily see that $\mu < 0$ if σ is sufficiently small. We remark that the same is true for the solution v_1 of Corollary 2.1.

3. Construction of the operator k^ε

In this section we will construct the operator k^ε solving equation (1.1) for u announced in the introduction. Thus, we assume in the rest of this section f satisfies assumptions (H1)–(H3) and denote by X the space of all elements $v \in C_r^{1,\alpha}(\overline{B})$ such that $v'(1) = 0$ endowed with its natural norm.

Let v_0 be a fixed element of X such that $v_0(\lambda_0) = v^*$ at a unique $\lambda_0 \in (0, 1)$. Here v^* is as in (H3). Further, we assume $v_0'(\lambda_0) \neq 0$.

We consider, for v on an X-neighborhood of v_0, the problem

$$\varepsilon^2 \Delta u = f(u, v) \quad \text{in } B, \tag{3.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B. \tag{3.2}$$

We also denote by $h(s)$ the function defind as

$$h(s) = \begin{cases} h_-(s) & \text{if } s < v^*, \\ h_+(s) & \text{if } s \geq v^*. \end{cases}$$

Our main purpose in this section is to establish the following results

Proposition 3.1. *There exist a neighborhood $\mathscr{N}$ in X of v_0, a number $\varepsilon_0 > 0$ and a family of continuous operators $k^\varepsilon : \mathscr{N} \to C^{2,\alpha}(\overline{B})$ defined for $0 < \varepsilon < \varepsilon_0$, such that*

(1) *for $v \in \mathscr{N}$, $0 < \varepsilon < \varepsilon_0$, the function $u = k^\varepsilon(v)$ solves* (3.1)–(3.2).

(2) $\lim_{\varepsilon \to 0} k^\varepsilon(v) = h(v)$ *uniformly on compacts of $\overline{B} \backslash \{|x| = \lambda\}$.*

More precisely, given $\rho > 0$ *there exist numbers* $M > 0$ *and* $\varepsilon_1 \in (0, \varepsilon_0)$ *such that for all* $0 < \varepsilon < \varepsilon_1$, $v \in \mathcal{N}$ *one has*

$$|k^\varepsilon(v)(r) - h(v(r))| \le \rho \quad \text{if } |r - \lambda(v)| \ge M\varepsilon,$$
$$\sup_{v \in \mathcal{N}, 0<\varepsilon<\varepsilon_0} \|k^\varepsilon(v)\|_{L^\infty(B)} < +\infty.$$

Therefore,

$$\sup_{v \in \mathcal{N}} \|k^\varepsilon(v) - h(v)\|_{L^q_r} \to 0 \quad \text{as } \varepsilon \to 0,$$

for any $1 \le q < \infty$.

The proof of Proposition 3.1 is based on the construction of a first approximation to a solution of (3.1)–(3.2) as given by Lemma 3.1 below. Before stating it, we observe that every v in a sufficiently small X-neighborhood of v_0 has the property that it takes the value v^* at a unique $\lambda = \lambda(v) \in (0, 1)$ such that $v'(\lambda) \neq 0$ and, moreover, λ depends continuously on v in the X-topology.

Lemma 3.1. *There exist a neighborhood* $\mathcal{N}$ *in* X *of* v_0, *a number* $\varepsilon_0 > 0$ *and a family of continuous operators* $\tilde{k}^\varepsilon : \mathcal{N} \to C^{2,\alpha}(\overline{B})$ *defined for* $0 < \varepsilon < \varepsilon_0$ *and such that*

(1) *For any* $v \in \mathcal{N}$, $0 < \varepsilon < \varepsilon_0$, *the function* $u_\varepsilon = \tilde{k}^\varepsilon(v)$ *satisfies an equation of the form*

$$\varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon, v) + \psi^\varepsilon(v) \quad \text{in } B, \tag{3.3}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial B, \tag{3.4}$$

where

$$\sup_{v \in \mathcal{N}} \|\psi^\varepsilon(v)\|_{L^\infty(B)} = o(\varepsilon). \tag{3.5}$$

Here, $\lim_{\varepsilon \to 0} o(\varepsilon)/\varepsilon = 0$.

(2) *Given* $\rho > 0$ *there exist a number* $M > 0$ *and* $\varepsilon_1 \in (0, \varepsilon_0)$ *such that for all* $0 < \varepsilon < \varepsilon_1$, $v \in \mathcal{N}$ *one has*

$$|\tilde{k}^\varepsilon(v)(r) - h(v(r))| \le \rho \qquad \text{if } |r - \lambda(v)| \ge M\varepsilon. \tag{3.6}$$

(3) $\sup_{v \in \mathcal{N}, 0<\varepsilon<\varepsilon_0} \|\tilde{k}^\varepsilon(v)\|_{L^\infty(B)} < +\infty$.

We will proceed assuming the validity of this result postponing its proof to the end of the section.

In what follows we will assume $v_0'(\lambda_0) > 0$. The opposite case can be dealt with in the same way as will become apparent from the proofs.

Let $\tilde{k}^\varepsilon$ be the operator in Lemma 3.1. We consider the following eigenvalue problem in $C^2_r(\overline{B})$.

$$\mathcal{L}^v_\varepsilon \phi \equiv \varepsilon^2 \Delta \phi - f_u(\tilde{k}^\varepsilon(v), v)\phi = \mu\phi \quad \text{in } B, \tag{3.7}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial B. \tag{3.8}$$

Let $\mu_1(\varepsilon, v) > \mu_2(\varepsilon, v) > \cdots$ denote the sequence of (radial) eigenvalues of this problem with associated eigenfunction $\phi_1, \phi_2, \ldots$. We need the following result.

Lemma 3.2. *There exist positive constants* ε_0, c, C *and a neighborhood* $\mathcal{N}$ *of* v_0 *such that for all* $0 < \varepsilon < \varepsilon_0$, $v \in \mathcal{N}$ *the following assertions hold.*

(a) *We have*

$$\lim_{\varepsilon \to 0} \frac{\mu_1(\varepsilon, v)}{\varepsilon} = cJ'(v^*)v'(\lambda(v)) \quad \textit{uniformly on } v \in \mathcal{N}.$$

Here J *is given by* (H3).

(b) 0 *is not an eigenvalue of* $\mathcal{L}_\varepsilon^v$. *Moreover,*

$$\|(\mathcal{L}_\varepsilon^v)^{-1} z\|_{L_r^\infty(B)} \leq C\|z\|_{L_r^\infty(B)}$$

for all z *such that* $\int_B z\phi_1(\varepsilon, v) = 0$.

Proof. Let us prove part (a). Fix a small bounded neighborhood $\mathcal{N}$ of v_0 in X. Consider sequences $\varepsilon_j \downarrow 0$, $v_j \in \mathcal{N}$ and denote $\mu_j = \mu_1(\varepsilon_j, v_j)$, $\lambda_j = \lambda(v_j)$. To establish (1) it suffices to show that, after passing to a suitable subsequence, we have

$$\lim_{j \to \infty} \left| \frac{\mu_j}{\varepsilon_j} - cJ'(v^*)v_j'(\lambda_j) \right| = 0. \tag{3.9}$$

Note that, since v_j is bounded in $C_r^{1,\alpha}$, we may assume, after passing to a subsequence, $v_j \to \overline{v}$ in the C^1-sense, where $\overline{v} \in C_r^1$ attains the value v^* at a unique $\overline{\lambda} \in (0, 1)$ and $\overline{v}'(\overline{\lambda}) > 0$. Thus (3.9) will follow if we show

$$\lim_{j \to \infty} \frac{\mu_j}{\varepsilon_j} = cJ'(v^*)\overline{v}'(\tilde{\lambda}) \tag{3.10}$$

with c a certain constant independent of $\overline{v}$.

Let ϕ_j denote a positive eigenfunction associated to μ_j normalized so that $\|\phi_j\|_{L^\infty} = 1$. We also denote $u_j = k_{\varepsilon_j}(v_j)$. Then ϕ_j satisfies

$$\varepsilon_j^2 \Delta\phi_j = (f_u(u_j, v_j) + \mu_j)\phi_j \quad \text{in } B, \tag{3.11}$$

$$\frac{\partial \phi_j}{\partial n} = 0 \quad \text{on } \partial B. \tag{3.12}$$

While u_j satisfies

$$\varepsilon^2 \Delta u_j = f(u_j, v_j) + \psi_j \quad \text{in } B, \tag{3.13}$$

$$\frac{\partial u_j}{\partial n} = 0 \quad \text{on } \partial B \tag{3.14}$$

where $\psi_j = \psi^{\varepsilon_j}(v_j)$ is as in Lemma 3.1.

For a radial function $p(r)$ we will use the notation $\tilde{p}_j(t)$ to designate the function $\tilde{p}_j(t) = p(\lambda_j + \varepsilon_j t)$. Observe that $\tilde{u}_j(t)$ satisfies the equation

$$\ddot{\tilde{u}}_j + \frac{\varepsilon_j(N-1)\dot{\tilde{u}}_j}{\lambda_j + t\varepsilon_j} = f(\tilde{u}_j, \tilde{v}_j) + \tilde{\psi}_j, \qquad t \in \left(\frac{-\lambda_j}{\varepsilon_j}, \frac{1-\lambda_j}{\varepsilon_j} \right). \tag{3.15}$$

Using the properties of u_j, ψ_j as given by Lemma 3.1 and a standard compactness argument, we find that, passing to a suitable subsequence, we may assume that $\tilde{u}_j \to \tilde{u}$ in the C^2-sense over compacts of the real line, where $\tilde{u}$ satisfies

$$\ddot{\tilde{u}} + f(\tilde{u}, v^*) = 0 \tag{3.16}$$

and $\tilde{u}(-\infty) = h_-(v^*)$, $\tilde{u}(+\infty) = h_+(v^*)$. It is well known that such a solution satisfies that $\dot{\tilde{u}}(t) > 0$ for all t and that both $\tilde{u}$ and $\dot{\tilde{u}}$ approach their limits exponentially as $t \to \pm\infty$.

Next, we claim that

$$\liminf_{j\to\infty} \mu_j \geq 0. \tag{3.17}$$

To show this we use the variational characterization of μ_j given by

$$\mu_j = -\inf\left\{ \frac{\varepsilon^2 \int_B |\nabla h|^2 + f_u(u_j, v_j)h^2}{\int_B h^2} | u \in H^1_r(B), u \neq 0 \right\}. \tag{3.18}$$

Using $h_j(r) = \dot{\tilde{u}}(\frac{r-\lambda_j}{\varepsilon_j})$ as a test function in (3.18) we easily see that

$$\liminf_{j\to\infty} \mu_j \geq -\int_{-\infty}^{\infty} (\ddot{\tilde{u}}(t)^2 + f_u(\tilde{u}(t), v^*)\dot{\tilde{u}}(t)^2)\, dt. \tag{3.19}$$

But this last integral equals zero, as follows from the fact that $\dot{\tilde{u}}$ satisfies on the real line the equation

$$\ddot{z} = f_u(\tilde{u}, v^*)z, \quad z(\pm\infty) = 0. \tag{3.20}$$

Thus (3.17) holds.

Now, from (3.11), (3.12), (3.17) and using the Maximum Principle we obtain that for sufficiently large j, ϕ_j maximizes on some interval of the form $[\lambda_j - \varepsilon_j M, \lambda_j + \varepsilon_j M]$ for some fixed constant M. On the other hand, $\tilde{\phi}_j(t) = \phi_j(\lambda_j + \varepsilon_j t)$ clearly satisfies

$$\ddot{\tilde{\phi}}_j + \frac{\varepsilon_j(N-1)\dot{\tilde{\phi}}_j}{\lambda_j + t\varepsilon_j} = (f_u(\tilde{u}_j, v_j) + \mu_j)\tilde{\phi}_j \quad \text{on } \left(-\frac{\lambda_j}{\varepsilon_j}, \frac{1-\lambda_j}{\varepsilon_j}\right). \tag{3.21}$$

It is easy to see, again applying the Maximum Principle to (3.11), (3.12), that μ_j must be bounded above. Let $\mu^* \geq 0$ be an accumulation point of μ_j. From (3.21), we see that, as before, we may assume that $\tilde{\phi}_j \to \tilde{\phi}$ in the C^2-sense over compacts of the real line, where $\tilde{\phi}$ is bounded, positive and satisfies on the real line:

$$\ddot{\tilde{\phi}} = (f_u(\tilde{u}, v^*) + \mu^*)\tilde{\phi}.$$

But since $\dot{\tilde{u}} > 0$ satisfies equation (3.20) and decays fast, we obtain, after an integration by parts,

$$\mu^* \int_{-\infty}^{\infty} \tilde{\phi}\dot{\tilde{u}} = 0$$

and hence $\mu^* = 0$. Moreover, a simple argument involving the wronskian of these functions shows that they are linearly dependent.

With no loss of generality, we will assume henceforth $\tilde{\phi} = \dot{\tilde{u}}$.

Let us set $w_j := \dot{\tilde{u}}_j$. Then w_j satisfies the equation

$$\dot{w}_j + \varepsilon_j \frac{(N-1)w_j}{\lambda_j + t\varepsilon_j} = f(\tilde{u}_j, \tilde{v}_j) + \tilde{\psi}_j. \tag{3.22}$$

Multiplying (3.22) by $\dot{\tilde{\phi}}_j$, integrating by parts between $-\delta/\varepsilon_j$ and δ/ε_j using (3.21), and then dividing the resulting equation by ε_j we obtain

$$\begin{aligned} &\frac{1}{\varepsilon_j}(w_j\dot{\tilde{\phi}}_j - f_j(\tilde{u}_j\tilde{v}_j)\tilde{\phi}_j)|_{-\delta/\varepsilon_j}^{\delta/\varepsilon_j} \\ &\qquad + 2(N-1)\int_{-\delta/\varepsilon_j}^{\delta/\varepsilon_j} \frac{w_j\dot{\tilde{\phi}}_j}{(\lambda_j+\varepsilon_j t)}\,dt - \frac{1}{\varepsilon_j}\int_{-\delta/\varepsilon_j}^{\delta/\varepsilon_j} \tilde{\psi}_j\dot{\tilde{\phi}}_j\,dt \\ &= \frac{\mu_j}{\varepsilon_j}\int_{-\delta/\varepsilon_j}^{\delta/\varepsilon_j} w_j\tilde{\phi}_j\,dt - \int_{-\delta/\varepsilon_j}^{\delta/\varepsilon_j} f_v(\tilde{u}_j, \tilde{v}_j)\tilde{v}_j'\tilde{\phi}_j\,dt. \end{aligned} \tag{3.23}$$

We will show that each term in the left-hand side of (3.23) approaches zero as $j \to \infty$. To do this, we need the following fact.

Claim. For a fixed and small $\delta > 0$ there exist positive constants M, η such that

$$|\tilde{\phi}_j(t)| + |\dot{\tilde{\phi}}_j(t)| \le \exp(-\eta|t|) \quad \text{for } M \le |t| \le \frac{\delta}{\varepsilon_j}.$$

Proof. Assume that M, δ are chosen so that

$$c_j(r) := f_u(\tilde{u}_j, r, l_j) + \mu_j \ge k > 0 \quad \text{for } M\varepsilon_j \le |r - \lambda_j| < 2\delta. \tag{3.24}$$

Consider the annulus $A_j := \{x | \lambda_j + M\varepsilon_j < |x| < \lambda_j + 2\delta\}$. Since ϕ_j satisfies

$$\varepsilon_j^2\Delta\phi_j = c_j(|x|)\phi_j \quad \text{in } A_j \tag{3.25}$$

and c_j satisfies (3.12), it follows from Lemma 3.3 in [22], see also [9, p. 230], that there is a number $\eta > 0$ such that

$$|\phi_j(|x|)| \le \exp\left(-\eta\frac{d(x)}{\varepsilon_j}\right) \quad \text{for } x \in A_j$$

where $d(x) = \operatorname{dist}(x, \partial A_j) = \min\{|x| - \lambda_j + M\varepsilon_j, \lambda_j + 2\delta - |x|\}$. Hence, if $M < t < \frac{\delta}{\varepsilon}$, we have $|\tilde{\phi}_j(t)| \le \exp(-\eta t + M)$. A similar estimate is found for $|\dot{\tilde{\phi}}_j(t)|$, for example using (3.25) and elliptic estimates. The same argument applies to obtain an estimate on $-\frac{\delta}{\varepsilon} < t < -M$, and the result of the claim follows. □

On the other hand, from (3.22) we see that

$$\frac{d}{dt}(w_j(\lambda_j + t\varepsilon_j)^{N-1}) = (\lambda_j + t\varepsilon_j)^{N-1}\{f(\tilde{u}_j, \tilde{v}_j) + \tilde{\psi}_j\}. \tag{3.26}$$

Since the amount between { } is uniformly bounded and so is w_y, we conclude

$$|w_j(t)| \le C(1 + |t|)$$

for some $C > 0$. Using this estimate and the exponential decay of $\tilde{\phi}_j$, $\dot{\tilde{\phi}}_j$, we obtain from (3.23) and the Dominated Convergence Theorem that the two first terms approach zero as $j \to \infty$. The same happens to the third term since $\|\psi_j\|_{L^\infty} = o(\varepsilon)$. Therefore, letting $j \to \infty$ in (3.23) and again using Dominated Convergence we conclude

$$\begin{aligned} \lim_{j\to\infty}\frac{\mu_j}{\varepsilon_j}\int_{-\infty}^{\infty}\dot{\tilde{u}}^2\,dt &= \overline{v}'(\overline{\lambda})\int_{-\infty}^{\infty} f_v(\tilde{u}, v^*)\dot{\tilde{u}}\,dt \\ &= \int_{h_-(v^*)}^{h_+(v^*)} f_v(s, v^*)\,ds = J'(v^*). \end{aligned}$$

Let $c = 1/\int_{-\infty}^{\infty} \dot{u}^2\,dt$. Note that this c does not depend on the function $\overline{v}$. We conclude therefore the validity of (3.9) with this number c. This concludes the proof of part (a).

Next we prove part (b). We need to prove the following: There exist a neighborhood $\mathcal{N}$ of v_0 in X and a positive number C such that, for all ε sufficiently small, any given $z \in C_r^2$ with $\int_B z\phi_1(\varepsilon, v) = 0$ and $w \in C_r^2$ satisfying

$$\varepsilon^2 \Delta w - f_u(\tilde{k}^\varepsilon(v), v)w = z \quad \text{in } B, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial B, \tag{3.27}$$

one has

$$\|w\|_{L^\infty} \le C\|z\|_{L^\infty}. \tag{3.28}$$

Let $\mathcal{N}$ be some small neighborhood of v_0. To prove this assertion we argue by contradiction: we assume the existence of sequences $\varepsilon_j \to 0$, $v_j \in \mathcal{N}$, $z_j \to 0$ in C_r^0 with $\int z\phi_j = 0$ and $w_j \in C_r^2$ such that $\|w_j\|_\infty = 1$ and

$$\varepsilon_j^2 \Delta w_j - f_u(u_j, v_j)w_j = z_j \quad \text{in } B, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial B. \tag{3.29}$$

Here, as before, we have denoted $u_j = \tilde{k}^\varepsilon(v_j)$ and $\phi_j = \phi_1(\varepsilon_j, v_j)$. Let us also set $\mu_j = \mu_1(\varepsilon_j, v_j)$. We observe that from (3.29) and the definition of μ_j one gets

$$\mu_j \int_B w_j\phi_j = \int_B z\phi_j = 0. \tag{3.30}$$

Assume that $\|w_j\|_\infty = w_j(s_j) = 1$. Note that since $z_j \to 0$ uniformly, (3.29) implies that $s_j \in [\lambda_j - M\varepsilon_j, \lambda_j + M\varepsilon_j]$ for some $M > 0$ and all j large enough.

Using the notation in the proof of part (a), we see that from (3.29) one gets, after passing to a suitable subsequence,

$$\tilde{w}_j(t) \to c\dot{u}(t)$$

uniformly on compacts of the real line for some $c > 0$. Recall that we may also assume

$$\tilde{\phi}_j(t) \to c'\dot{u}(t)$$

for some $c' > 0$. Moreover,

$$|\tilde{\phi}_j(t)| \le \exp(-\eta|t|) \quad \text{for } |t| > M,$$

and some constants M, $\eta > 0$. On the other hand, from (3.30) we see that

$$\int_{-\lambda_j/\varepsilon_j}^{1-\lambda_j/\varepsilon_j} \tilde{w}_j(t)\tilde{\phi}_j(t)(\lambda_j + \varepsilon_j t)^{N-1}\,dt = 0.$$

Hence, using Dominated Convergence we find

$$\int_{-\infty}^{\infty} \dot{u}^2 = 0$$

which is obviously a contradiction. This concludes the proof of part (b). □

Remark 3.1. If in part (a) we had assumed $v_0'(\lambda_0) < 0$ instead of $v_0'(\lambda_0) > 0$, we would have obtaind

$$\lim_{\varepsilon \to 0} \frac{\mu_1(v, \varepsilon)}{\varepsilon} = c'|v'(\lambda(v))|J'(v^*) \tag{3.31}$$

where now $c' = 1/\int_{-\infty}^{\infty} \dot{\overline{u}}^2$ and $\overline{u}$ is any solution of

$$\ddot{\overline{u}} = f(\overline{u}, v^*)$$

such that $\overline{u}(-\infty) = h_+(v^*)$, $\overline{u}(\infty) = h_-(v^*)$.

Note, on the other hand that part (b) implies, in particular, the existence of a $k > 0$ such that $\mu_2(v, \varepsilon) \le -k$ for all $v \in \mathscr{N}$ and ε small. This last fact implies the validity of the statement in part (b) of the lemma with L^2-norms replacing the L^∞-norms, since the operator $\mathscr{L}_\varepsilon^v$ is selfadjoint.

We can now proceed to the proof of Proposition 3.1.

Proof of Proposition 3.1. Decompose $u = u^\varepsilon + t\phi_1(v, \varepsilon) + w$ where $u^\varepsilon = \tilde{k}^\varepsilon(v)$, $\int_B w\phi_1(v, \varepsilon) = 0$, $\phi_1 > 0$ and $\|\phi_1(v, \varepsilon)\|_\infty = 1$. Then we can rewrite equation (3.1), (3.2) as the following equivalent system.

$$w + (\mathscr{L}_\varepsilon^v)^{-1} P\{F_\varepsilon^v(w + t\phi_1) - \psi(\varepsilon, v)\} = 0, \tag{3.32}$$

$$t + \frac{1}{\mu_1(v, \varepsilon)} \int_B (F_\varepsilon^v(w + t\phi_1) - \psi(\varepsilon, v)) \frac{\phi_1}{\|\phi_1\|_{L_r^2}} = 0. \tag{3.33}$$

Here we have denoted

$$Pz = z - \left(\int_B z\phi_1\right) \frac{\phi_1}{\|\phi_1\|_{L_r^2}}$$

and

$$F_\varepsilon^v(z) = f(u^\varepsilon + z, v) - f(u^\varepsilon, v) - f_u(u^\varepsilon, v)z.$$

Note that there is a constant $C > 0$ such that

$$\|F(z)\|_\infty \le C\|z\|_\infty^2 \tag{3.34}$$

and

$$\|F(z_1) - F(z_2)\|_\infty \le C\delta\|z_1 - z_2\|_\infty \tag{3.35}$$

for all $z, z_1, z_2 \in B(0, \delta)$, the ball center 0, radius δ in C_r^0, and any $\delta > 0$.

First we solve equation (3.32) for $w \in W^\varepsilon$ where

$$W^\varepsilon = \left\{w \in C_r^0 | \int_B w\phi_1 = 0\right\}.$$

Call $T(\varepsilon, v, t, w)$ the second summand in (3.32). Observe that from Lemma 3.2 (b) and (3.34) we get

$$\|T(\varepsilon, v, t, w)\|_\infty \le C\{\|w\|_\infty^2 + t^2 + o(\varepsilon)\}. \tag{3.36}$$

Hence, there is a $\delta > 0$ such that T, regarded as an operator in w, applies the ball $\overline{B}(0, \delta) \cap W^\varepsilon$ into itself provided that $|t| \le \delta$, ε is sufficiently small and $v \in \mathscr{N}$. Similarly, now using (3.35) and reducing δ if necessary we obtain that this operator is a contraction. Therefore, (3.32) possesses a unique solution $w = w(\varepsilon, t, v)$ in $\overline{B}(0, \delta) \cap W^\varepsilon$ which clearly depends continuously on its arguments. Moreover, w is Lipschitz in t, uniformly on $v \in \mathcal{N}$ and small ε. Also, (3.36) yields

$$\|w(\varepsilon, t, v)\|_\infty \leq Ct^2 + o(\varepsilon). \tag{3.37}$$

Substituting this w into (3.33), we obtain the equation in t:

$$t + \frac{1}{\mu_1(v, \varepsilon)} \int_B (F_\varepsilon^v(w(\varepsilon, t, v) + t\phi_1) - \psi(\varepsilon, v)) \frac{\phi_1}{\|\phi_1\|_{L_r^2}} = 0. \tag{3.38}$$

Call $\alpha(t, \varepsilon, v)$ the second summand in (3.38). Then we have, using Lemma 3.2 part (a) and (3.34),

$$|\alpha(t, \varepsilon, v)| \leq C\{t^2 + o(\varepsilon)\} \frac{1}{\|\phi_1\|_{L_r^2}} \frac{1}{\varepsilon} \int_B |\phi_1|. \tag{3.39}$$

Now, letting $\tilde{\phi}_1(\varepsilon, v)(t) = \phi_1(\varepsilon, v)(\lambda(v) + \varepsilon t)$, we see that

$$\frac{1}{\varepsilon} \int_B |\phi_1| = \int_{-\lambda(v)/\varepsilon}^{(1-\lambda(v))/\varepsilon} \tilde{\phi}_1(\varepsilon, v)(t)(\lambda(v) + \varepsilon t)^{N-1}\, dt.$$

Now, arguing as in the proof of Lemma 3.2 part (a), we see that $\tilde{\phi}_1(\varepsilon, v)(t) \to \dot{\tilde{u}}(t)$ as $\varepsilon \to 0$, uniformly on compacts, where $\tilde{u}$ satisfies

$$\ddot{\tilde{u}} = f(\tilde{u}, v^*)$$

and $\tilde{u}(-\infty) = h_-(v^*)$, $\tilde{u}(\infty) = h_+(v^*)$. Also, recalling that $\tilde{\phi}_1(\varepsilon, v)(t)$ has a uniform exponential decay, we get from the Dominated Convergence Theorem

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_B \phi_1(\varepsilon, v) = \int_{-\infty}^{\infty} \dot{\tilde{u}} = h_+(v^*) - h_-(v^*) \tag{3.40}$$

and this convergence is uniform on $v \in \mathcal{N}$. Similarly, we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{1/2}} \|\phi_1(\varepsilon, v)\|_{L_r^2} = \left(\int_{-\infty}^{\infty} |\dot{\tilde{u}}|^2\right)^{1/2} \tag{3.41}$$

uniformly on $\mathcal{N}$.

From (3.39), (3.40) and (3.41) we see that

$$|\alpha(t, \varepsilon, v)| \leq C\left\{\frac{t^2}{\varepsilon^{1/2}} + \varepsilon^{1/2}\theta(\varepsilon)\right\}, \tag{3.42}$$

where $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let us fix a number $1/2 < \eta < 1$ and assume $|t| \leq \theta(\varepsilon)^\eta \varepsilon^{1/2}$. Then, from (3.42) we obtain

$$|\alpha(t, \varepsilon, v)| \leq C\{\theta(\varepsilon)^{2\eta} + \theta(\varepsilon)\}\varepsilon^{1/2} \leq \theta(\varepsilon)^\eta \varepsilon^{1/2},$$

if ε is sufficiently small. Therefore, α as a function of t applies the interval $|t| \leq \theta(\varepsilon)^\eta \varepsilon^{1/2}$ into itself for all small ε.

Now, recalling that w is uniformly Lipschitz on t and using (3.35), we easily obtain that α becomes a contraction on t in this range for all small ε. Hence, we have a unique solution $t = t(\varepsilon, v)$ to the equation (3.38) such that $|t| \leq \theta(\varepsilon)^\eta \varepsilon^{1/2}$.

We conclude that

$$k^\varepsilon(v) = \tilde{k}^\varepsilon(v) + t(\varepsilon, v)\phi_1(\varepsilon, v) + w(t(\varepsilon, v), \varepsilon, v)$$

solves (3.1), (3.2). Moreover,

$$\|k^\varepsilon(v) - \tilde{k}^\varepsilon(v)\|_\infty = o(\varepsilon^{1/2})$$

and, from the properties of $\tilde{k}^\varepsilon(v)$, we easily deduce the desired properties of $k^\varepsilon(v)$. This concludes the proof of the proposition. □

It only remains to prove Lemma 3.1. We need the following preliminary result.

Lemma 3.3. *There exist a neighborhood $\mathcal{N}$ of v_0, a number $\varepsilon_0 > 0$ and families of radial solutions to* (3.1), (3.2), $\{h_+(\varepsilon, v)\}_{0<\varepsilon<\varepsilon_0}$, $\{h_-(\varepsilon, v)\}_{0<\varepsilon<\varepsilon_0}$, $v \in \mathcal{N}$, *which define continuous operators $\mathcal{N} \to C_r^2$ and such that*

$$\sup_{v\in\mathcal{N}} \|h_\pm(\varepsilon, v) - h_\pm(v)\|_{L^\infty} = o(\varepsilon), \tag{3.43}$$

$$\varepsilon \sup_{v\in\mathcal{N}} \|\nabla h_\pm(\varepsilon, v) - \nabla h_\pm(v)\|_{L^\infty(B)} = o(\varepsilon), \tag{3.44}$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. We will prove the existence of $h_-(\varepsilon, v)$ as in the statement of the theorem. The proof for $h_+(\varepsilon, v)$ is the same. Fix a small neighborhood $\mathcal{N}$ of v_0 and set $w = u - h_-(v)$. Then, since $\frac{\partial v}{\partial n} = 0$ on ∂B, (3.1), (3.2) can be rewritten as

$$\varepsilon^2 \Delta w = f(h_-(v) + w, v) - \varepsilon^2 \Delta h_-(v) \quad \text{in } B, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } B \tag{3.45}$$

where $\Delta h_-(v)$ is understood in the distributional sense (recall that v is only $C^{1,\alpha}$). We consider, for $t \in [0, 1]$, the auxiliary problem

(3.46)
$$\varepsilon^2 \Delta w = t(f(h_-(v) + w, v) - \varepsilon^2 \Delta h_-(v)) + (1-t) f_u(h_-(v), v) w \quad \text{in } B, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial B.$$

Fix a small number $\rho_0 > 0$. We will prove the following fact.

Claim. Let

$A_\varepsilon = \{w \in C_r^{0,1} \mid \|w\|_{L^\infty} \le \rho_0$ and w solves (3.46) for some $t \in [0, 1]$, $v \in \mathcal{N}\}$.

Then,

$$\sup_{w\in A_\varepsilon} \|w\|_{L^\infty} = o(\varepsilon)$$

where $o(\varepsilon)/\varepsilon \to 0$ to $\varepsilon \to 0$.

Proof. Assume the contrary, i.e. the existence of sequences $v_n \to \tilde{v}$ in C^1, $v_n \in \mathcal{N}$, $\varepsilon_n \to 0$, $t_n \to \bar{t} \in [0, 1]$ and solution w_n to (3.46) for $v = v_n$, $\varepsilon = \varepsilon_n$, $t = t_n$ such that, for some $c > 0$,

$$\frac{\varepsilon_n}{\|w_m\|_\infty} \le c.$$

We assume that, for some $r_n \in [0, 1]$, $w_n(r_n) = \|w_n\|_\infty$ (the case $w_n(r_n) = -\|w_n\|_\infty$ is similar). Let us assume $r_n \to \bar{r} \in [0, 1]$. We further consider two subcases.

(a) $r_n < \mu\varepsilon_n$, for some $\mu > 0$.
(b) $\lim_{n\to\infty} \frac{r_n}{\varepsilon_n} = +\infty$.

Assume first (a) holds. For a function $P(r)$, we set $\widetilde{P}(s) = P(\varepsilon_n s)$. Further, let us define $z_n(s) = \tilde{w}_n/\|w_n\|_\infty$. Then

$$\Delta z_n = \frac{1}{\|w_n\|_\infty} t_n(f(h_-(\tilde{v}) + \tilde{w}_n, \tilde{v}_n)) - \frac{1}{\|w_n\|_\infty}\Delta h_-(\tilde{v}_n) + (1-t_n)f_u(h_-(\tilde{v}_n), \tilde{v}_n)z_n \quad \text{in } B(0, 2\mu),$$

in the distributional sense. Note that

$$\frac{1}{\|w_n\|_\infty}\Delta h_-(\tilde{v}_n) = \operatorname{div}\theta_n$$

where $\theta_n = (\varepsilon_n/\|w_n\|_\infty)\widetilde{\nabla v_n}$. Observe that θ_n is uniformly bounded. It follows, from the elliptic estimates in Chapter 8 of [13], that we may assume $z_n \to z$ in the $C^{1,\alpha}$ sense. Moreover, we may also assume θ_n converges in the C^1 sense to a constant. Hence z is actually of class C^2, and we may also assume $\tilde{w}_n$ converges in the C^1 sense to some $\tilde{w}$ with $\|\tilde{w}\|_\infty \le \rho_0$. Thus, z satisfies

$$\Delta z - c(s)z = 0 \quad \text{in } B(0, 2\mu) \tag{3.47}$$

where

$$c(s) = \tilde{t}\int_0^1 f_u(h_-(\overline{v}(0)) + \tau\tilde{w}(s), \overline{v}(0))\,d\tau + (1-t)f_u(h_-(\overline{v}(0)), \overline{v}(0)).$$

Note that $c(s) > 0$ if ρ_0 was chosen small enough. But, since we are in case (a), we see that z has a positive maximum in $B(0, 2\mu)$; hence (3.47) is impossible because of the Maximum Principle. This discards case (a).

Now, if (b) holds, similar arguments applied to $\tilde{w}_n(s) = w_n(r_n + \varepsilon_n s)$ lead us to the following situation: there are functions $\tilde{w}(s)$, $z(s)$ which satisfy

$$\ddot{z} - \left(\bar{t}\int_0^1 f_u(h_-(\overline{v}(\bar{r})) + \tau\tilde{w}, \overline{v}(\bar{r}))\,d\tau + (1-\bar{t})f_u(h_-(\overline{v}(\bar{r})), \overline{v}(\bar{r}))\right) z = 0$$

on $(-\infty, 0]$, where $\|\tilde{w}\|_\infty \le \rho_0$, $z'(0) = 0$ and z maximizes at 0. This is again impossible and concludes the proof of the claim. □

From the claim, existence of a family of solutions $h_-(\varepsilon, v)$ satisfying (3.1), (3.2) and (3.42) follows from a simple degree-theoretical argument applied to the family of equations (3.46). On the other hand, an indirect argument similar to the one employed in the proof of the claim, with the aid of (3.45) and elliptic estimates, shows assertion (3.44). Finally, continuity of the family h_- in its arguments is an immediate consequence of the uniqueness of the solution w of (3.45) with $\|w\|_\infty \le \rho_0$. This last fact follows easily from the Maximum Principle. This concludes the proof of the lemma. □

Proof of Lemma 3.1. We will construct the desired approximation $\tilde{k}^\varepsilon(v)$. We fix $v \in \mathscr{N}$, where $\mathscr{N}$ is some small neighborhood of v_0 in X. Let $h_+(\varepsilon, v)$, $h_-(\varepsilon, v)$ be the families predicted by Lemma 3.3. We use the notation

$$h_\pm^\varepsilon = h_\pm(\varepsilon, v), \quad h_\pm^0 = h_\pm(v).$$

Dependence on v should always be understood implicit in the different notation we will use in the course of this proof.

Let us also set $a_\varepsilon = h_-^\varepsilon$, $b_\varepsilon = h_+^\varepsilon - h_-^\varepsilon$, $a = h_-^0$, $b = h_+^0 - h_-^0$ and rewrite in (3.1), (3.2) $u = a^\varepsilon + b^\varepsilon w$. Then, (3.1) becomes

$$(3.48)\quad \varepsilon^2(\Delta a_\varepsilon + w\Delta b_\varepsilon) + \varepsilon^2 b_\varepsilon w'' + (N-1)\varepsilon^2 \frac{b_\varepsilon w'}{r} + 2\varepsilon^2 b_\varepsilon' w' = f(a_\varepsilon + w b_\varepsilon, v).$$

Also, for a function $p(r)$ we will denote, as usual, $\tilde{p}(t) = p(\lambda(v) + \varepsilon t)$. Then, (3.48) rewritten in terms of $\tilde{w}$ becomes

$$(3.49)\quad \varepsilon^2(\Delta \tilde{a}_\varepsilon + \tilde{w}\Delta \tilde{b}_\varepsilon) + \tilde{b}_\varepsilon \ddot{\tilde{w}} + (N-1)\varepsilon \frac{\tilde{b}_\varepsilon \dot{\tilde{w}}}{\lambda(l) + \varepsilon t} + 2\varepsilon \tilde{b}_\varepsilon' = f(\tilde{a}_\varepsilon + \tilde{w}\tilde{b}_\varepsilon, \tilde{v}).$$

We look for a formal approximation to $\tilde{w}$ which, using an idea of Hale and Sakamoto [14], we take of the form

$$(3.50)\qquad z_\varepsilon(t) = z_0(t) + \varepsilon z_1(t).$$

Then, from (3.49), z_0, z_1 should respectively satisfy the equations

$$(3.51)\qquad b_* \ddot{z}_0 = g(z_0, v^*)$$

and

$$(3.52)\qquad b_* \ddot{z}_1 + t b_*' \ddot{z}_0 + (N-1)\frac{b_*' z_0'}{\lambda} = g_u(z_0, v^*) z_1 + g_v(z_0, v^*) v_*' t$$

and the conditions at infinity

$$(3.53)\qquad z_0(+\infty) = 1, \quad z_0(-\infty) = 0, \quad \dot{z}_0(\pm\infty) = 0,$$

$$(3.54)\qquad z_1(\pm\infty) = 0, \quad \dot{z}_1(\pm\infty) = 0.$$

Here we have denoted

$$g(w, v) = f(h_-(v) + (h_+(v) - h_-(v))w, v).$$

The argument applied in [14, p. 372] then yields the existence of unique z_0, z_1 satisfying (3.51)–(3.54). Moreover, z_0, z_1 have exponential decay estimates

$$(3.55)\qquad |z_0(t)| \le \exp(-k|t|) \quad \text{for } t < -M$$

and

$$(3.56)\qquad |z_0(t)| \le \exp(-k|t|) \quad \text{for } t > M.$$

Analogous estimates hold for $\dot{z}_0$, z_1 and $\dot{z}_1$. Next we fix some small number $\delta > 0$ and set

$$u_\varepsilon(r) = h_\varepsilon^-(r) + \zeta_-(r)\left\{\zeta_+(r) z_\varepsilon\left(\frac{r-\lambda}{\varepsilon}\right) + (1 - \zeta_+(r))\right\}(h_+^\varepsilon - h_-^\varepsilon).$$

Here ζ_-, ζ_+ are C^∞ cut-off functions such that for all $v \in \mathcal{N}$

$$\zeta_-(r) = \begin{cases} 0 & \text{if } 0 < r < \lambda(v) - 2\delta, \\ 1 & \text{if } r > \lambda(v) - \delta, \end{cases}$$

$$\zeta_+(r) = \begin{cases} 1 & \text{if } r < \lambda(v) + \delta, \\ 0 & \text{if } r > \lambda(v) + 2\delta. \end{cases}$$

Thus,

$$u_\varepsilon = \begin{cases} h_-^\varepsilon & \text{if } r < \lambda(v) - 2\delta\,, \\ h_+^\varepsilon & \text{if } r > \lambda(v) + 2\delta\,, \\ z^\varepsilon(\frac{r-\lambda}{\varepsilon}) & \text{if } |\lambda - \lambda(v)| < \delta. \end{cases}$$

$\tilde{k}^\varepsilon(v) \equiv u_\varepsilon$ is the approximation we are looking for. Let $\psi_\varepsilon = \varepsilon^2\Delta u^\varepsilon - f(v^\varepsilon, v)$. We need to show that ψ_ε satisfies estimate (3.5).

Note that $\psi_\varepsilon = 0$ if $|r - \lambda(v)| > 2\delta$. Now, let us assume

$$\lambda(v) - 2\delta < r < \lambda(v) - \delta. \tag{3.57}$$

Then

$$u^\varepsilon(r) = h_-^\varepsilon + \zeta_-(r) z_\varepsilon\left(\frac{r-\lambda}{\varepsilon}\right)(h_+^\varepsilon - h_-^\varepsilon)$$

and

$$\psi_\varepsilon = f(h_-^\varepsilon, v) - f\left(h_-^\varepsilon + \zeta_- z_\varepsilon\left(\frac{r-\lambda}{\varepsilon}\right)(h_+^\varepsilon - h_-^\varepsilon), v\right) + \varepsilon^2\Delta\left(\zeta_-(r) z_\varepsilon\left(\frac{r-\lambda}{\varepsilon}\right)(h_+^\varepsilon - h_-^\varepsilon)\right);$$

since $z_\varepsilon(t)$ and its two first derivatives are exponentially small for t large and negative, we conclude

$$|\psi_\varepsilon(r)| = O(e^{-\beta/\varepsilon})$$

for some $\beta > 0$, uniformly on r satisfying (3.57) and on v. The same clearly happens if $\lambda(v) + \delta < r < \lambda(v) + 2\delta$. Hence, in particular,

$$|\psi_\varepsilon(r)| = o(\varepsilon) \quad \text{if } |r - \lambda(v)| \geq \delta.$$

Next we consider the case $|r - \lambda(v)| \leq \delta$. In this range we have $u^\varepsilon = a_\varepsilon + z_\varepsilon b_\varepsilon$. For a family of functions $f_\varepsilon(t, v)$ we will write $f_\varepsilon(t, v) = o(\varepsilon)$ to designate the fact that $\sup_v \|f_\varepsilon\|_\infty = o(\varepsilon)$.

From Lemma 3.3 we know that

$$h_\pm^\varepsilon - h_\pm^0 = o(\varepsilon), \quad \varepsilon(\nabla h_\pm^\varepsilon - \nabla h_\pm^0) = o(\varepsilon), \quad \varepsilon^2\Delta h_\pm = o(\varepsilon).$$

From this and the growth properties of z_0 and z_1 we easily obtain that

$$\varepsilon^2\tilde{\Delta}u^\varepsilon = o(\varepsilon) + \tilde{b}\ddot{z}_0 + (N-1)\frac{\tilde{b}\varepsilon\dot{z}_0}{\lambda + \varepsilon t} + 2\varepsilon\tilde{b}'\dot{z}_0 + \varepsilon\tilde{b}\ddot{z}_1 + (N-1)\frac{\varepsilon^2\dot{z} - 1}{\lambda + \varepsilon t} + 2\varepsilon^2\tilde{b}'\dot{z}_1.$$

Note that the last two terms are $o(\varepsilon)$. From this and the definitions of z_0 and z_1, we obtain

$$\varepsilon^2\tilde{\Delta}u^\varepsilon = o(\varepsilon) + \left(\tilde{b} - \frac{\varepsilon t b_*'\tilde{b}}{b_*}\right)\ddot{z}_0 + (N-1)\varepsilon b\dot{z}_0\left(\frac{1}{\lambda + \varepsilon t} - \frac{1}{\lambda}\right) + 2\varepsilon\left(\tilde{b}' - \frac{\tilde{b}}{b_*}b_*'\right)\dot{z}_0 + \varepsilon\frac{b}{b_*}\{g_u(z_0, v^*)z_1 + g_v(z_0, v^*)v_*'t\}.$$

Now, using the exponential decay of $\dot{z}_0$, $\ddot{z}_0$ and $\dot{z}_1$ and the above equality we get

$$\varepsilon^2\tilde{\Delta}u^\varepsilon = o(\varepsilon) + b_*\ddot{z}_0 + \varepsilon\{g_u(z_0, v^*)z_1 + g_v(z_0, v^*)v_*'t\}.$$

Hence, $\psi_\varepsilon = \varepsilon^2\Delta u^\varepsilon - f(u^\varepsilon, v)$ satisfies in this range $\tilde{\psi}_\varepsilon = o(\varepsilon) - g_\varepsilon$, where

$$g_\varepsilon = f(\tilde{a}^\varepsilon + (z_0 + \varepsilon z_1)\tilde{b}^\varepsilon, \tilde{v}) - g(z_0, v^*) - g_w(z_0, v^*)\varepsilon z_1 + g_v(z_0, v^*)v'_*\varepsilon t.$$

Observe that since $a^\varepsilon - a = o(\varepsilon)$ and $b^\varepsilon - b = o(\varepsilon)$, we have

$$f(\tilde{a}^\varepsilon + (z_0 + \varepsilon z_1)\tilde{b}^\varepsilon, \tilde{v}) = f(\tilde{a} + \tilde{b}(z_0 + \varepsilon z_1), \tilde{v}) + o(\varepsilon).$$

But $f(\tilde{a} + \tilde{b}(z_0 + \varepsilon z_1), \tilde{v}) = g(z_0 + \varepsilon z_1, \tilde{v})$, by definition of g. Note that

$$g(z_0 + \varepsilon z_1, \tilde{v}) = g(z_0, \tilde{v}) + g_w(z_0, \tilde{v})\varepsilon z_1 + o(\varepsilon),$$

since z_1 is bounded. Thus

$$\begin{aligned} g_\varepsilon = {} & (g(z_0, \tilde{v}) - g(z_0, v^*) - g_w(z_0, v^*)\varepsilon z_1 + g_v(z_0, v^*)v'_*\varepsilon t) \\ & + (g_w(z_0, \tilde{v}) - g_w(z_0, v^*))\varepsilon z_1 + o(\varepsilon) = \mathrm{I} + \mathrm{II} + o(\varepsilon). \end{aligned}$$

But $g(s, \tilde{v}) = 0$ if $s \to 0, 1$. Since z_0 approaches exponentially these numbers as $t \to -\infty$ and $t \to \infty$ respectively, it follows that $\mathrm{I} = o(\varepsilon)$. We have that $\mathrm{II} = o(\varepsilon)$ too. Indeed, note that

$$|g_w(z_0, \tilde{v}) - g_w(z_0, v^*)|\varepsilon|z_1| \le |\tilde{v} - v^*| \le C\varepsilon^{1+\alpha} \sup |tz_1(t)| = o(\varepsilon).$$

Note that the constant C above depends on $\mathcal{N}$ but not on v. Hence $\psi_\varepsilon = o(\varepsilon)$ if $|r - \lambda(v)| \le \delta$, as desired. The other properties of the approximation $k^\varepsilon(v)$ stated in the lemma follow immediately from the construction. This concludes the proof. □

Remark 3.2. A by-product of the above construction is the stability or instability of the family $k^\varepsilon(v)$. Indeed, formula (3.31) still holds for these solutions and hence the family will be stable for small ε if $J'(v^*) < 0$ and unstable if $J'(v^*) > 0$. In the former case, a simpler proof of Proposition 3.1 can be given with the aid of the direct method of the calculus of variations. In fact, in this case the desired family can be captured as global minimizers of the associated energy functional. We do not give details of this construction here but only remark that it can be carried out following the ideas in [1] and [2] where related scalar problems under radial symmetry were treated. See also [7] for a degree-theoretical construction of these "stable layers" when no radial symmetry is assumed.

However, $J'(v^*) < 0$ is not generally expected in applications to systems. In particular, it does not hold in our application to the Gierer and Meinhardt model.

4. Main result

We can now state and prove our main result.

Theorem 4.1. *Fix $\sigma \le \sigma_0$ (or, equivalently, $D \ge 1/\sigma_0$) with σ_0 as in Proposition 2.2. Let v_0 be the decreasing solution to* (2.1) *predicted by Proposition* 2.1 *for $\theta = v^*$. Then there exists an $\varepsilon_0 > 0$ and a family of radial solutions $\{(u_\varepsilon, v_\varepsilon)\}_{0<\varepsilon<\varepsilon_0}$ to problem* (P) *such that*

(1) *$v_\varepsilon \to v_0$ as $\varepsilon \to 0$ in the $C^{1,\alpha}$-sense.*

(2) *$u_\varepsilon \to h(v_0)$ as $\varepsilon \to 0$ uniformly of compacts of $\overline{B}\backslash\{|x| = \lambda_0\}$. More precisely, given $\rho > 0$ there exist numbers $M > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$ such that, for all $0 < \varepsilon < \varepsilon_1$, one has*

$$|u_\varepsilon(r) - h(v_\varepsilon(r))| \le \rho \quad \text{if } |r - \lambda_\varepsilon| \ge M\varepsilon.$$

Here, $h = h^{v^*}$ *where* h^θ *is given by* (1.7). λ_ε *is the unique radius such that* $v_\varepsilon(\lambda_\varepsilon) = v^*$.

(3) $$\sup_{0<\varepsilon<\varepsilon_0} \|u_\varepsilon\|_{L^\infty(B)} < +\infty.$$

A similar statement holds true replacing v_0 *with* v_1 *given by Corollary* 2.1.

Proof. For notational simplicity we will assume $\sigma = 1$ during the proof. Let $\mathscr{N}$ be an X-neighborhood of v_0 as given in Proposition 3.1. Then for $v \in \mathscr{N}$ we can solve for u (1.1) into the form $u = k^\varepsilon(v)$. Substituting this u into (1.2) we obtain the single equation

(4.1) $$\Delta v = g(k^\varepsilon(v), v) \quad \text{in } B, \qquad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial B.$$

Since the operator k^ε is continuous on $\mathscr{N}$ regarded as an operator from X into L^q_r, we obtain that the operator $G^\varepsilon : X \to L^q_r$ given by

$$G^\varepsilon(v)(r) := g(k^\varepsilon(v)(r), v(r))$$

is also continuous. Moreover, it is easily seen to be compact.

We will show that (4.1) has a solution for all sufficiently small ε by proving

(4.2) $$\deg(I - T_\varepsilon, \mathscr{N}, v_0) \neq 0$$

for a small X-neighborhood $\mathscr{N}$ of v_0 where

$$T_\varepsilon(v) := (\Delta - I)^{-1}(G^\varepsilon(v) - v).$$

To prove (4.2) we will use Proposition 2.2 together with the invariance of the Leray-Schauder degree under compact homotopies. Consider for $t \in [0, 1]$ the homotopy

$$Q_t(v) := (\Delta - I)^{-1}(G^\varepsilon(v) - v) + t(\Delta - I)^{-1}(G^\varepsilon(v) - \tilde{G}(v))$$

where $\tilde{G}$ is as in Proposition 2.2. Then, $Q_0 = T$, $Q_1 = T_\varepsilon$ where T is given by (2.12). For $v \in \mathscr{N}$ we have

$$\begin{aligned} \|v - Q_t(v)\|_X &\geq \|v - T(v)\|_X - \|(\Delta - I)^{-1}\|_{\mathscr{L}(L^q_r, X)} \|G^\varepsilon(v) - \tilde{G}(v)\|_{L^q_r} \\ &\geq \inf_{v \in \partial\mathscr{N}} \|v - T(v)\|_X - c\|(\Delta - I)^{-1}\|_{\mathscr{L}(L^q_r, X)} \|k^\varepsilon(v) - h(v)\|_{L^q_r} \end{aligned}$$

where in the last inequality we have used that G may be assumed to be Lipschitz. But from Proposition 3.1 part (3), we may reduce $\varepsilon > 0$ so that

$$\sup_{v \in \partial\mathscr{N}} \|k^\varepsilon(v) - h(v)\|_{L^q_r} < \inf_{v \in \partial\mathscr{N}} \|v - T(v)\|_X / c\|(\Delta - I)^{-1}\|_{\mathscr{L}(L^q_r, X)}.$$

Hence, for small ε, $\|v - Q_t(v)\|_X > 0$ for all $v \in \partial\mathscr{N}$ and all $t \in [0, 1]$. We conclude that Q_t is an admissible homotopy between T and T_ε. Thus

$$\deg(I - T_\varepsilon, \mathscr{N}, v_0) = \deg(I - T, \mathscr{N}, v_0).$$

Since the latter degree is nonzero, thanks to Proposition 2.2, (4.2) follows, thus proving the existence of a solution v_ε to (4.1) in $\mathscr{N}$. Hence, letting $u_\varepsilon := k^\varepsilon(v_\varepsilon)$, we find that $(u^\varepsilon, v^\varepsilon)$ solves problem (P).

We shall next prove (1). Since v_ε and $G^\varepsilon(v_\varepsilon)$ are uniformly bounded, $v_\varepsilon = (\Delta - I)^{-1}(G^\varepsilon(v_\varepsilon) - v_\varepsilon)$ and $(\Delta - I)^{-1}$ maps compactly L^∞ into $C_r^{1,\alpha}$, we see that $\{v_\varepsilon\}$ is precompact in $C_r^{1,\alpha}$. We will show that $v_\varepsilon \to v_0$ in $C_r^{1,\alpha}$.

Let $\tilde{v}$ be any accumulation point of v_ε in $C_r^{1,\alpha}$ and select a sequence $\varepsilon_j \to 0$ such that $v_{\varepsilon_j} \to \tilde{v}$ in $C_r^{1,\alpha}$. Then $\tilde{v} \in \mathscr{N}$. If $\mathscr{N}$ was chosen sufficiently small, we see that $\tilde{v}(r)$ takes the value v^* at just one r. In particular, it follows that

$$h(v_{\varepsilon_j}) \to h(\tilde{v}) \quad \text{in } L_r^q \tag{4.3}$$

for any $q > 1$. On the other hand

$$\begin{aligned} &\|G^{\varepsilon_j}(v_{\varepsilon_j}) - \widetilde{G}(\tilde{v})\|_{L_r^q} \\ &\quad \le \|g(k^{\varepsilon_j}(v_{\varepsilon_j}, v_{\varepsilon_j}) - g(h(v_{\varepsilon_j}), v_{\varepsilon_j})\|_{L_r^q} \\ &\qquad + \|g(h(v_{\varepsilon_j}), v_{\varepsilon_j}) - g(h(\tilde{v}), \tilde{v})\|_{L_r^q} \\ &\quad \le c\left\{\sup_{v \in \mathscr{N}} \|k^{\varepsilon_j}(v) - h(v)\|_{L_r^q} + \|h(v_{\varepsilon_j}) - h(\tilde{v})\|_{L_r^q}\right\}. \end{aligned} \tag{4.4}$$

From Proposition 3.1 part (3) and (4.3), we see that the right-hand side of (4.4) tends to zero as $j \to \infty$. Now, from (4.4) and the fact that $v_{\varepsilon_j} = (\Delta - I)^{-1}(G(v_{\varepsilon_j}) - v_{\varepsilon_j})$ we obtain $v_{\varepsilon_j} \to \tilde{v}$ in X and

$$\tilde{v} = (\Delta - I)^{-1}(\widetilde{G}(\tilde{v}) - \tilde{v}). \tag{4.5}$$

Since v_0 is an isolated solution of (4.1), it follows from (4.5) that $\tilde{v} = v_0$. Hence $v_\varepsilon \to v_0$ in $C^{1,\alpha}$ as $\varepsilon \to 0$. The above proof also gives this convergence in X, as desired. The other parts of the theorem follow easily from the properties of k^ε given in Proposition 3.1. This concludes the proof. □

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